Math 876: PDE Seminar 2022
Week 5: Illustrations

February 24, 2022

General: In §3.8 we go over some more examples of C_0-semigroups, covering even more examples in the homework. In §3.9 we see that small perturbations of sectorial operator produces a sectorial operator (for an appropriate notion of small).

Due to the President's day holiday, this week we only meet on Thursday (not on Tuesday). Office hours this week are Tuesday/Wednesday from 12:30pm-2:00pm.

Required Reading: §3.8.1- §3.8.4, §3.9

Supplementary Reading: Appendix C.1 Bochner integrals.

Important Concepts: ∂_{xx} operator on $\mathbb{R}$, Leray projection; Stokes operator; wave equation; perturbations of sectorial operators.

Reading Questions: No required reading questions this week. But as always, if you have any questions feel free to talk to me or your classmates about them!

Presentations: See HW.

Problems:

Heat equation on a line: Do problem # 13 (p23) from *Geometric Theory of Semilinear Parabolic Equations* by D. Henry, repeated below.

Let $X = C(\mathbb{R}, \mathbb{R})$, the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the sup-norm. Let $Au(x) = -\partial_{xx}u(x)$ for

$$u \in \mathcal{D}(A) = \left\{u \in C^2(\mathbb{R}, \mathbb{R}) : u, u', u'' \in X\right\},$$

and let $X_1 = Cl_X(\mathcal{D}(A))$. (Note: $X_1 \neq X$). If $\lambda = m^2$, $Re(m) > 0$, calculate

$$(\lambda I + A)^{-1} f(x) = \frac{1}{2m} \int_{-\infty}^{\infty} e^{-m|x-\xi|} f(\xi) d\xi,$$

for $f \in X_1$. Prove A is sectorial in X_1, $\sigma(A) = [0, +\infty)$, and for any complex $\lambda \neq 0$ with $|arg\lambda| \leq \theta < \pi$, and any $f \in X_1$,

$$\|(\lambda I + A)^{-1} f\|_X \leq |\lambda|^{-1} \sec\theta/2 \, \|f\|_X.$$

Show that, for any $\phi \in X_1$ (i.e., any uniformly continuous $\phi \in X$) if $u(x,t) = (e^{-At}\phi)(x)$ with $t > 0$, $x \in \mathbb{R}$, then u, $\partial_t u$, $\partial_{xx} u$ are continuous and $\partial_t u = \partial_{xx} u$ for $t > 0, -\infty < x < \infty$.

Also $u(x,t) \to \phi(x)$ as $t \to 0^+$, uniformly in x. (i.e. show that u is a classical solution on $[0,\infty)$.) Can you prove this is the only (classical) solution of

$$\begin{aligned} \partial_t u &= \partial_{xx} u \qquad (t > 0) \\ u(0^+, x) &= \phi(x), \qquad -\infty < x < \infty. \end{aligned}$$

Linear Boussinesq equation: Do problem 38.10 from the book. Can you say something more about the case when $\Omega = (0,\pi)^2 \subseteq \mathbb{R}^2$?

Stokes Equation: Consider the Stokes equations on $\Omega = (0,\pi)^2$ given by

$$\begin{aligned} \partial_t - \nu \triangle u + \nabla p &= f, \\ \nabla \cdot u &= 0 \end{aligned}$$

taken with Dirichlet boundary conditions. Using a Fourier basis, compute the Helmholtz/Leray projection $\mathbb{P}$ and the Stokes operator A. What is $\sigma(A)$?

Hint: For $m \leq n$, fix a m dimensional subspace $V \subseteq \mathbb{R}^n$, and fix a matrix $A \in \mathbb{R}^{n\times m}$ such that V is spanned by the columns of A. Then the projection $P_V : \mathbb{R}^n \to V$ has a matrix representation given by $P_V x = A(A^T A)^{-1} A^T x$ and is independent of the matrix A.