Math 876: PDE Seminar 2022
Week 6: Variation of constants

March 1 & 3, 2022

General:

We are now entering the most important chapter of the book! To begin the chapter, in Section 4.2 we consider the linear, non-autonomous equation

$$\partial_t u + Au = f(t),$$

and we define three different solution concepts: *mild*, *strong*, and *classical*. As one would expect: classical $\implies$ strong $\implies$ mild. In finite dimensional ODEs these solution concepts largely coincide, but not so in abstract evolutionary equations! In particular, we need to be mindful of both the temporal and spatial regularity of solutions (eg is f in L^1? L^∞? Is it continuous/Lipshitz/differentiable?) Often if we assume f has more temporal regularity then we can prove u has more spatial regularity. In the homework we explore this interplay for a specific example.

Required Reading: §4.1 through §4.2.2

Supplementary Reading: Appendix C.2

The Cantor–Lebesgue function is the quintessential example of where the Newton-Leibniz formula breaks down. This is remedied by introducing the concept of absolute continuity. Appendix C.2 generalizes this to Banach spaces, and while it isn't mentioned in the book,[1] but many of the theorems here (eg the second half of Lemma C.5) also apply if we take as a Banach space $W = \ell^1$.

Important Concepts: variation of constants; mild, strong, and classical solutions; Leibniz formula.

Reading Questions: Email me at least 3 questions on the reading at least an hour before class on Tuesday.

Presentations:

Solutions in the C_0 Theory (Tu): Start by briefly introducing the Variation of Constants formula, and the definitions for mild solutions and strong solutions. Then present the Leibniz formula and the proof of Lemma 42.5.

Solutions in the Analytic Theory (Tu): Briefly discuss Standing Hypotheses A & B, and give familiar examples of something satisfying Hypothesis B, and something satisfying A but not B. Then present the proof of Lemma 42.7. (For example, give) If

[1] For a reference see: Diestel, J., & Uhl, J. J. (1976). The Radon-Nikodym theorem for Banach space valued measures. *The Rocky Mountain Journal of Mathematics*, 6(1), 1-46.

time allows, discuss Lemmas 42.8 and how our results improve if we assume the stronger hypotheses in Theorem 42.9.

Notation:

Recall our definition of $\ell_s^2 \equiv \ell_{s,1}^2$ from Week 2 given as

$$\ell_s^2 = \left\{ \{a_k\}_{k\in\mathbb{Z}} : a_k \in \mathbb{C}, \|a\|_{\ell_s^2} < \infty \right\}, \qquad \|a\|_{\ell_s^2} = \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |a_k|^2 \right)^{1/2},$$

where we define weights

$$\langle k\rangle = (1 + |k|^2)^{1/2}. \tag{1}$$

Another option would be to define our weights as

$$\langle k\rangle = \begin{cases} 1 & \text{if } k = 0 \\ |k| & \text{if } k \neq 0. \end{cases} \tag{2}$$

The weights in (1) and (2) produce equivalent norms on ℓ_s^2. Since the weights from (2) are a bit nicer to work with, lets use them for this week. It will also be useful to recall that for the the Fourier transform defines an isomorphism between the the Sobolev spaces $H^s(\mathbb{T})$ and ℓ_s^2, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the torus. As a special case then we have $L^2(\mathbb{T}, \mathbb{R}) \cong \ell_0^2$.

The Sawtooth Function:

Consider the (discontinuous) sawtooth function $S : \mathbb{T} \to \mathbb{R}$ defined by

$$S(x) = 1 - \frac{x}{\pi}.$$

This function is in $L^2(\mathbb{T}, \mathbb{R})$, and may be given by the Fourier series

$$S(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(kx),$$

which converges in L^2. One can check that $S \in H^s(\mathbb{T}, \mathbb{R})$ for $s < 1/2$. The functions we will be looking at in the subsequent problems will look quite similar to the sawtooth function.

Problems:

Bochner Integrals, Part 1: Here we define a forcing function $f : [0, +\infty) \to \ell_0^2$ that we will use in the rest of the problems. We define f component-wise for $k \in \mathbb{Z}$ by

$$f_0(t) = 0, \qquad f_k(t) := \begin{cases} i\dfrac{k}{|k|} & \text{if } 0 < t < \dfrac{1}{|k|^{3/2}}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The function f essentially corresponds to a sine series, as $Re(f(t)) = 0$ and $f_{-k}(t) = (f_k(t))^*$.

NOTE: For the rest of the problems, we will fix $T = +\infty$.

(a) Prove that $f : [0,T) \to \ell^2_0$ is Bochner integrable.

(b) Calculate an upper bound on $\int_0^\infty \|f(t)\|_{\ell^2_0} dt$.

(c) Show that $f \in L^1_{loc}[0,T;\ell^2_0) \cap L^\infty_{loc}(0,T;\ell^2_0)$.

Solutions for the C_0-theory

Define the linear operator A on $L^2(\mathbb{T},\mathbb{R})$ by

$$Au = -\partial_x u$$

Note that $\mathcal{D}(A)$ is isomorphic to the Sobolev space $H^1(\mathbb{T},\mathbb{R}) \cong \ell^2_1$. Define the function $\tilde{f} \in L^1_{loc}[0,T;L^2(\mathbb{T},\mathbb{R}))$ by

$$\tilde{f}(t,x) = \sum_{k\in\mathbb{Z}} e^{ikt} f_k(t) e^{ikx},$$

and consider the PDE

$$u_t + Au = \tilde{f}(t). \tag{4}$$

For $k \in \mathbb{Z}$ we define a sequence of function $\{u_k\}_{k\in\mathbb{Z}} : [0,T) \to \ell^2_0$ as follows: For $k = 0$ we define $u_0(t) = 0$; for $k \geq 1$ we define

$$u_k(t) = \begin{cases} ie^{ikt} t & \text{if } 0 \leq t < k^{-3/2} \\ ie^{ikt} k^{-3/2} & \text{if } k^{-3/2} \leq t; \end{cases}$$

and for $k \leq -1$ we define $u_k(t) = (u_{-k}(t))^*$. Furthermore, define $u : [0,T) \to L^2(\mathbb{T},\mathbb{R})$ by

$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t) e^{ikx}.$$

(a) Show that $u(t)$ is the mild solution of (4) in the space L^2 with initial data $u(0) = u_0 = 0 \in H$. Can you say anything about the regularity of u with respect to t?

(b) Show that $u : [0,T) \to L^2(\mathbb{T},\mathbb{R})$ is not a strong solution. What does this tell us about the Leibniz Formula (42.7)?

(c) Show that $u : [0,T) \to H^s(\mathbb{T},\mathbb{R}) \cong \ell^2_s$ is a strong solution if $s < 0$.

Bochner Integrals, Part 2:

(Optional) For the last problem, it will be useful to know, for various values of p and s, when $f \in L^p_{loc}[0,T;\ell^2_s)$. To this end, let us define

$$h_s(t) := \begin{cases} \left|1 - t^{-\frac{2}{3}(1+2s)}\right|^{1/2} & \text{if } s \neq -1/2 \\ \sqrt{-\log(t)} & \text{if } s = -1/2. \end{cases}$$

One can show that for all $s \in \mathbb{R}$, exists a constant $C_s > 1$ such that

$$\frac{1}{C_s}\big(h_s(t) - 1\big) < \|f(t)\|_{\ell^2_s} < C_s\big(h_s(t) + 1\big) \qquad \forall t \in (0,1].$$

Use this to prove the following:

(a) If $1 \leq s$ then $f \notin L^1_{loc}[0,T;\ell^2_s)$ and $f \in L^\infty_{loc}(0,T;\ell^2_s)$.

(b) If $-1/2 < s < 1$ and $1 \leq p < \frac{3}{(1+2s)}$ then $f \in L^p_{loc}[0,T;\ell^2_s) \cap L^\infty_{loc}(0,T;\ell^2_s)$

(c) If $s = -1/2$ then $f \in L^p_{loc}[0,T;\ell^2_s)$ for all $1 \leq p < \infty$.

(d) If $s < -1/2$ then $f \in L^\infty_{loc}[0,T;\ell^2_s)$.

Solutions for the analytic theory:

Define the linear operator A on $L^2(\mathbb{T},\mathbb{R})$ by

$$Aw = -\partial_{xx} w.$$

Note that $L^2(\mathbb{T},\mathbb{R}) \cong \ell^2_0$ and we have the family of fractional interpolation spaces $\mathcal{D}(A^\alpha) \cong H^{2\alpha}(\mathbb{T},\mathbb{R}) \cong \ell^2_{2\alpha}$, e.g. $\mathcal{D}(A^{1/2}) \cong \ell^2_1$. In a slight abuse of notation, define the function $f : [0,+\infty) \to L^2(\mathbb{T},\mathbb{R})$ by

$$f(t,x) = \sum_{k\in\mathbb{Z}} f_k(t) e^{ikx}$$

and consider the nonautonomous heat equation on a torus:

$$w_t + Aw = f(t) \tag{5}$$

Define $w : [0,+\infty) \to L^2(\mathbb{T},\mathbb{R})$ as

$$w(t,x) = -2\sum_{k=1}^{\infty} w_k(t)\sin(kx),$$

where

$$w_k(t) := \begin{cases} \dfrac{1-e^{-k^2 t}}{k^2} & \text{if } 0 \leq t < \dfrac{1}{k^{3/2}} \\ e^{-k^2 t}\dfrac{\left(e^{\sqrt{k}}-1\right)}{k^2} & \text{if } \dfrac{1}{k^{3/2}} < t. \end{cases}$$

(a) Show that w is a mild solution of (5) in the space L^2 with initial condition $w(0) = 0$.

(b) Show that $w \in C[0,T;H^1) \cap C^{0,\theta_0}_{loc}(0,T;H^1)$ for some Hölder constant $1/100 < \theta_0$.

(c) Show that the mild solution of (5) with any initial condition $u_0 \in H^{-1}$ is in fact a strong solution in the space H^{-1}.