Math 876: PDE Seminar 2022
Week 7: Linear Skew Product Semiflow

March 15 & 17, 2022

General:

This week we are studying linear skew product semiflows in §4.3 and non-autonomous linear evolutionary equations in §4.4. This material can be seen to generalize Floquet theory from ODEs (see examples below). Or at least the beginnings of how to generalize Floquet theory. We will come back to skew product systems when studying exponential dichotomies and invariant manifolds.

The goal in §4.4, studying non-autonomous linear evolutionary equations, is not unreasonable. However the section has some really heavy notation. This stems from two problems we want to overcome:

(I) We want to allow for linear operators $B : V^{2\beta} \to W$ which lose regularity.

(II) We want to work with nonautonomous linear operators $B(t)$ that don't necessarily have a limit (or other nice properties) as $t \to \pm\infty$.

To solve problem (I) we use the fundamental theorem of Sectorial operators and a souped-up version of Gronwall's inequality. To solve problem (II) we introduce weaker notions of convergence through Fréchet spaces.

Primary Reading: §4.3 and §4.4.

Secondary Reading:

- Appendix A.7 - Fréchet Space
- Appendix D - The Gronwall-Henry inequality. It will also be helpful to look up some properties of the Γ function.
- §4.2.3 - §4.2.4 These sections cover *weak solutions*, which are related to the limit of ODE solutions one obtains via a sequence of Galerkin approximations.

Important Concepts: linear skew product semiflow; cocycle identity; the solution operator $\Phi(B, t)$; Fréchet Spaces; the different topologies on L^∞ and $\mathcal{M}^\infty$; the Gronwall-Henry inequality; the EX property.

Reading Questions: Email me at least 3 questions on the reading at least an hour before class on Tuesday.

Presentations:

Fréchet spaces (Tu):

In §4.4 we consider the $L^\infty(\mathbb{R};\mathcal{L}(V^{2\beta},W))$, but we impose different topologies, such as $\mathcal{T}_\infty$, $\mathcal{T}_{bo}$ and $\mathcal{T}_A$. The first topology makes L^∞ a Banach space, but the later two are Fréchet spaces. Tell us about what Fréchet spaces, why we shouldn't be afraid of them, and how to think of these different topologies. In §4.4.2 we introduce $\mathcal{M}^\infty$ and even more topologies. If time remains, talk about the continuity of the shift operator σ.

Gronwall-Henry Inequality (Tu):

Tell us about the Gronwall-Henry inequality and how it is used. In particular, fill in the details for how we get the estimate (44.8), and how we show Φ is Lipschitz continuous in v and B on p178-179. If you have time, talk about why you think the EX property so named.

Examples:

Floquet theory example from ODEs

Consider the ordinary differential equation $\dot{X} = f(X)$ for $X = (x, y)$ given as below:

$$\begin{aligned} \dot{x} &= x - y - x(x^2+y^2) \\ \dot{y} &= x + y - y(x^2+y^2) \end{aligned} \tag{1}$$

One can verify that $\gamma(t) := \{\cos t, \sin t\}$ is a periodic solution to (1). An initial condition $X(0) = \gamma(t_0) + h_0$ will have a solution given by $X(t) = \gamma(t) + h(t)$. If we drop the $\mathcal{O}(|h|^2)$ terms in (1) we obtain the non-autonomous linear system: $h_t = B(t)h$ where

$$B(t) = Df(\gamma(t)) = \begin{pmatrix} -2\cos^2(t) & -1 + -2\sin(t)\cos(t) \\ 1 - 2\sin(t)\cos(t) & -2\sin^2(t) \end{pmatrix}.$$

The monodromy matrix $\Phi(t)$ for this system may be computed to be: $\Phi(t) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}$. In general, even in ODEs, it is quite non-trivial to solve for the monodromy matrix.

Fundamental Theorem of Sectorial Operators

Consider the sectorial operator $A = -\partial_{xx}$ defined on $W = L^2([0,\pi],\mathbb{R})$ with Dirichlet 0 boundary conditions, whereby $\sigma(A) = \{k^2\}_{k=0}^\infty$ and $\|e^{-At}\|_{\mathcal{L}(W,W)} \leq e^{-t}$. Furthermore, let $V^{2\alpha}$ denote the fractional power spaces generated by A, and note that $V^{2\alpha} \cong H^\alpha([0,\pi],\mathbb{R}) \cong \ell^2_\alpha$.

By item (2) of the fundamental theorem of sectorial operators, for any $\beta \geq 0$ there is a constant $M_\beta > 0$ such that

$$\|A^\beta e^{-At}\|_{\mathcal{L}(W,W)} \leq M_\beta t^{-\beta} e^{-at} \qquad \forall t > 0, \tag{2}$$

where $a \in \mathbb{R}$ is some number for which (36.2) is satisfied.

In Problem II on Burger's equation we are particularly interested in this bound for the case $\beta = 1/2$. In this case the LHS of (2) becomes

$$\|A^{1/2}e^{-At}\|_{\mathcal{L}(W,W)} = \sup_{c\in\ell^2;\|c\|=1} \left(\sum_{k\in\mathbb{Z}} k e^{-k^2 t}|c_k|^2 \right)^{1/2} = \sup_{k\in\mathbb{Z}} \left| k e^{-k^2 t} \right| \tag{3}$$

If we fix $t > 0$ and pretend that k is a continuous variable, then $ke^{-k^2 t}$ will have a local maximum when $\frac{d}{dk} ke^{-k^2 t} = e^{-k^2 t}(1 - 2k^2 t) = 0$; the RHS of (3) is maximized when $k = (2t)^{-1/2}$. Plugging this in we obtain:

$$\|A^{1/2} e^{-At}\|_{\mathcal{L}(W,W)} \leq \begin{cases} (2et)^{-1/2} & \text{if } 0 < t < 1/2 \\ e^{-t} & \text{if } 1/2 \leq t. \end{cases}$$

If we choose $a = 1/2$ and $M_{1/2} = 2/3$, then the bound in (2) will be satisfied.

Problems:

1. Linear Skew Product Semiflow

Fix $f : C^{1,1}_{loc}(\mathbb{R}^n \to \mathbb{R}^n)$ and consider the ODE $u_t = f(u)$. Define $\varphi : \mathbb{R} \times M \to M$ as the flow generated by $u_t = f(u)$, and suppose that $M \subseteq \mathbb{R}^n$ is a compact invariant set. For each $m \in M$ define the $\Phi(m,t)$ as the unique solution to

$$\begin{aligned} \dot{\Phi}(m,t) &= Df(\varphi(t,m))\Phi(m,t) \\ \Phi(m,t) &= I \end{aligned}$$

where $I \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is the identity matrix. The matrix $\Phi(m,t)$ is sometimes called a monodromy matrix, or the principal fundamental matrix solution. Note then that the function

$$\pi(w,m,t) = (\Phi(m,t)w, \varphi(t,m)).$$

defines linear skew product semiflow on $\mathbb{R}^n \times M$

(a) For each $m \in M$ define the function $B(m;\cdot) : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ by

$$B(m;t) = Df(\varphi(t,m)). \tag{4}$$

Show that for all $m_0 \in M$ we have

$$B(m_0;\cdot) \in L^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)) \cap C^{0,1}(\mathbb{R}; \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$$

Hence, the association $m \mapsto B(m,\cdot)$ defines a map $\beta : M \to \mathcal{M}^\infty \subseteq L^\infty$ where $\beta(m) = B(m;\cdot)$.

(b) Consider the differential equation

$$x' = x - x^3 \tag{5}$$

and define $M = [-1,1]$, the global attractor. Show that if we are using the $\mathcal{T}_\infty$ topology on L^∞, then the map $\beta : M \to L^\infty$ is not continuous at $0 \in M$.

(c) Suppose $M \subseteq \mathbb{R}^n$ is an arbitrary compact invariant set, and consider $\mathcal{M}^\infty \subseteq L^\infty$ with the $\mathcal{T}_{bo}$ topology. Show that if we are using the $\mathcal{T}_{bo}$ topology on L^∞, then the map $\beta : M \to L^\infty$ is continuous.[1]

Conclude that the image $\beta(M) = \mathcal{K} \subseteq \mathcal{M}^\infty$ is compact, and that $\mathcal{K}$ is invariant under the flow σ defined in §4.4.2.

[1]Recall that if $f : M \to \mathbb{R}^n$ and $Lip(f) = \kappa$, then $\|\varphi(t,x_0) - \varphi(t,y_0)\| \leq \|x_0 - y_0\| e^{\kappa|t|}$ for all $x_0, y_0 \in M$ and $t \in \mathbb{R}$.

(d) Show that the function

$$\pi\left(v_0, B, \tau\right) = \left(\Phi(B,\tau)v_0, B_\tau\right)$$

defines a linear skew product semiflow on $\mathbb{R}^n \times \mathcal{K}$.

2. Burger's Equation

Consider the sectorial operator $A = -\partial_{xx}$ defined on $W = L^2([0,\pi],\mathbb{R})$ with Dirichlet 0 boundary conditions, whereby $\sigma(A) = \{k^2\}_{k=0}^{\infty}$ and $\|e^{-At}\|_{\mathcal{L}(W,W)} \leq e^{-t}$. Furthermore, let $V^{2\alpha}$ denote the fractional power spaces generated by A, and note that $V^{2\alpha} \cong H^\alpha([0,\pi],\mathbb{R}) \cong \ell^2_\alpha$.

In this problem we consider the forced, viscous Burger's equation:

$$u_t + Au = u_x u + f(t) \tag{6}$$

for some $f \in C^{1,1}(\mathbb{R}, W)$. Furthermore, suppose there exists some $b \in C^{0,1}(\mathbb{R}, C^1([0,\pi]))$ which is a globally defined solution to (6).

(a) For each $t \in \mathbb{R}$, define the map $B : V^1 \to W$ by

$$[B(t)h](y) = b_x(t,y)h(y) + b(t,y)h_x(y) \tag{7}$$

where $h \in V^1$, $y \in [0,\pi]$ and $b_x = \partial_x b$. Show that

$$B \in L^\infty(\mathbb{R}; \mathcal{L}(V^1, W)) \cap C^{0,1}(\mathbb{R}; \mathcal{L}(V^1, W))$$

and thereby $B \in \mathcal{M}^\infty$. Furthermore, show that

$$\|B\|_{L^\infty(\mathbb{R};\mathcal{L}(V^1,W))} \leq \|b\|_{L^\infty(\mathbb{R},C^1([0,\pi]))}.$$

(b) For $h_0 \in W$, consider the initial condition $u(0) = b(0) + h_0$ to (6). After some cancellation and dropping the higher order terms in (6), we obtain the linearized equation about b below:

$$h_t - \partial_{xx} h = B(t)h, \qquad h(0) = h_0. \tag{8}$$

Show that there is a unique strong solution $h \in C^{0,\frac{1}{2}}(0,\infty; V^1)$ solving (8) a.e. in W.

(c) For the functions $E_{r,c}(z)$ defined in Appendix D, show that

$$e^z \leq E_{\frac{1}{2},1}(z) \leq (1+\sqrt{z})e^z$$
$$\sqrt{\pi z}e^z \leq E_{\frac{1}{2},\frac{1}{2}}(z) \leq \sqrt{\pi z}(1 + e^z + ze^z).$$

(d) Calculate μ in terms of $\|B\|_\infty$, for μ given by equation (44.8) in the book.

(e) Show that if $\|b\|_{L^\infty(\mathbb{R},C^1([0,\pi]))} < \frac{3}{2\sqrt{2\pi}}$ then $\lim_{t\to\infty} \|h(t)\|_{V^1} = 0$ for any $h_0 \in W$.