# Math 876: PDE Seminar 2022
# Week 8: Nonlinear Theory

March 22 & 24, 2022

**General:**

We now have the tools to prove theorems about the local existence of solutions to nonlinear PDEs, global existence vs the blowup alternative, and patching together individual solutions to define a semiflow. These results can be seen as generalizations of familiar theorems from ODEs, and the proofs often follow the same overall approach.

The differences needed for the PDE theory are in the finer details: we have different types of solutions (mild vs strong vs classical) so we'll need stronger assumptions to obtain stronger regularity; in the analytic theory we have several different norms to keep track of, and bound using the Gronwall-Henry inequality; when dealing with non-autonomous problems we want to define a skew-product semi-flow, and this introduces some weird topologies.

But at the end of the day we get some pretty strong results. We are able to identify systems where global existence is assured, and the Herculean theorem shows that with Standing Hypothesis A, an invariant set of mild solutions are in fact classical!

**Primary Reading:** §4.6 and §4.7.

**Secondary Reading:**

- C.3 Fréchet differentiability is the gold standard of differentiability.
- *Problems #2 and #3 look at the numerical method of exponential integrators. If you are interested in learning more see the reference below.*

  Hochbruck, M., & Ostermann, A. (2010). Exponential integrators. *Acta Numerica*, 19, 209-286.

**Important Concepts:** the spaces $C^k_{Lip}$; local existence and uniqueness of solutions; global existence and blowup alternative; construction of nonlinear semiflow;

**Reading Questions:** Email me at least 3 questions on the reading at least an hour before class on Tuesday.

**Presentations:**

**Construction of Semiflow (Tu):** Tell us about §4.6.3 and §4.6.4 on the construction of the nonlinear semiflow. How does Theorem 46.4 get used?

**Herculean Theorem (Tu):** Tell us about Theorem 47.6 and its proof. Why do you think this will be an important theorem? Explain why, without extra hypotheses, we don't get that $u$ is a mild solution in the space $V^2$ (cf Theorem 42.10).

**Problems:**

**Banach Algebra**

The book begins its study of the nonlinear theory with a discussion of the nonlinearities to be considered. In particular Lipschitz continuous and Fréchet differentiable functions.

**Definition 1.** *A* ***Banach algebra*** *is a Banach space $X$ with a multiplication operation $* : X \times X \to X$ that satisfies*

$$
\begin{aligned}
x * (y * z) &= (x * y) * z \\
(x + y) * z &= x * z + y * z, \qquad x * (y + z) = z * y + x * z \\
\alpha(x * y) &= (\alpha x) * y = x * (\alpha y) \\
\|x * y\| &\leq \|x\| \|y\|,
\end{aligned}
$$

*for all $x, y, z \in X$ and all scalars $\alpha$.*

The Banach algebra is commutative if $x * y = y * x$ for all $x, y \in X$. A unit element $e$ of a Banach algebra $X$, if one exists, satisfies $x * e = e * x = x$ for all $x \in X$.

**Examples:**

- If $W$ is a Banach space, then the space of bounded linear operators $\mathcal{L}(W, W)$ is a noncommutative Banach algebra, where $*$ is given by composition of maps.
- If $\Omega \subseteq \mathbb{R}^n$, then $C^0(\Omega, \mathbb{R})$ is a Banach algebra, where $*$ is given by pointwise multiplication of functions. The space $X_1 = \{f \in C^0(\Omega, \mathbb{R}) : f|_{\partial\Omega} = 0\}$ is an example of Banach algebra without a unit element.
- If $s \geq 0$ then $\ell^1_{s,d}$ is a Banach algebra. For $a, b \in \ell^1_{s,d}$ the product $a * b$ is the discrete convolution, given componentwise by $(a * b)_k = \sum_{k_1 + k_2 = k} a_{k_1} b_{k_2}$ for $k_1, k_2, k \in \mathbb{Z}^d$.

**Theorem 1.** *Let $W$ be a Banach algebra with product $*$ and fix $c \in W$. Define maps $F, G : W \to W$ by*

$$
F(w) = c * w, \qquad G(w) = w * w.
$$

*Then $F$ and $G$ are Fréchet differentiable on $W$, and*

$$
DF(w)h = c * h, \qquad DG(w)h = w * h + h * w.
$$

*for all $h \in W$.*

## #1. Banach Algebra Problem

Suppose $W$ is a commutative Banach algebra over $\mathbb{R}$, with product $*$ and unit $e$. For $n \in \mathbb{N}$ we recursively define $w^n$ by $w^0 := e$ and $w^{n+1} = w * w^n$.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a real analytic function. That is, there is a sequence $\{c_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ for which $f$ is given by

$$
f(x) = \sum_{k=0}^{\infty} c_k x^k,
$$

which converges absolutely for all $x \in \mathbb{R}$. We define the function $F : W \to W$ by

$$
F(w) = \sum_{k=0}^{\infty} c_k w^k.
$$

(a) Show that $F \in C^1_F(W,W)$. Moreover, show that the Fréchet derivative $DF(w) \in \mathcal{L}(W,W)$ of $F$ at $w \in W$ is given by

$$DF(w)h = \left( \sum_{k=1}^{\infty} k c_k w^{k-1} \right) * h,$$

for all $h \in W$. Then show that $DF \in C^0(W, \mathcal{L}(W,W))$.

(b) Use the Mean Value Formula (93.4) on page 619 to show that $F \in C^1_{Lip}(W,W)$.

## The Exponential Euler's method

At the beginning of the semester, we took a look at how to numerically solve PDEs with the Fourier transform and the Euler method. Unless the time step is very small, numerical solutions will blow up when using the explicit Euler method. The implicit Euler method solved this spurious numerical blowup problem, but small time steps are still needed for accuracy. For nonlinear PDEs we used an IMEX method: treating the linear part implicitly and the nonlinear part explicitly. This week we take a look at the exponential Euler's method and try to answer the question: Why should our numerical approximation be close to the true mathematical solution?

Fix $(e^{-At}, -A)$ a $C_0$-semigroup on a Banach space $W$, and let $F \in C^1_{Lip}(W,W)$. Fix an initial condition $u_0 \in W$ and consider the initial value problem

$$\partial_t u + Au = F(u(t)), \qquad u(0) = u_0.$$

The mild solution of the differential equation will satisfy the variation of constants formula:

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} F(u(s)) ds.$$

If the only information we have is the initial condition $u(0) = u_0$, then the simplest thing we could do to approximate the solution is plug $u(s) = u_0$ into the integral. This leads us to the exponential Euler approximation:

$$\bar{u}(t) := e^{-At}u_0 + t\varphi_1(-At)F(u_0), \tag{1}$$

where $\varphi_1$ is the entire, analytic function

$$\varphi_1(z) := \frac{e^z - 1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}.$$

We define $\varphi_1(z)$ more generally below:

**Definition 2.** *Fix $(e^{-At}, -A)$ a $C_0$-semigroup on a Banach space $W$.*

*(i) For $t > 0$, define:* $\varphi_1^i(-At) := \frac{1}{t}\int_0^t e^{-A(t-s)} ds$ .

*(ii) Suppose $A$ is invertible. For $t > 0$ define:* $\varphi_1^{ii}(-At) := (-At)^{-1}\left(e^{-At} - I\right)$.

*(iii) Suppose $A \in \mathcal{L}(W,W)$. For $t > 0$, define:* $\varphi_1^{iii}(-At) := \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(-At)^n$.

## #2. Equivalence of Definitions

(Optional) This problem shows that the various definitions of $\varphi_1^i, \varphi_1^{ii}, \varphi_1^{iii}$ all agree when they overlap. Hence we can simply refer to the function $\varphi_1$ without ambiguity.

(a) Use Theorem 31.4 (2) to show that $\varphi_1^i(-At) = \varphi_1^{ii}(-At)$ whenever $A$ is invertible.

(b) (Operator Calculus) Suppose that $A \in \mathcal{L}(W,W)$ and consider the differential equation on $\mathcal{L}(W,W)$ given below:

$$\partial_t X + AX = I, \qquad X(0) = 0 \in \mathcal{L}(W,W) \tag{2}$$

(i) Show that $X^i(t) = t\varphi_1^i(-At)$ is a mild solution to (2).

(ii) Show that $X^{iii}(t) = t\varphi_1^{iii}(-At)$ is a classical solution to (2).

(iii) Use Lemma 42.1 to conclude that $\varphi_1^i(-At) = \varphi_1^{iii}(-At)$.

## #3. Convergence of the exponential Euler method:

It follows from the previous problem, that for our definition of $\bar{u}(t)$ in (1), we have

$$\begin{aligned}\bar{u}(t) &= e^{-At}u_0 + t\varphi_1(-At)F(u_0)\\ &= e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u_0)ds.\end{aligned}$$

In this problem we estimate how close $\bar{u}$ is to the true solution. Throughout, we assume the same hypotheses and notation as in Theorem 46.1 of Sell & You.

(a) Show that for any $\epsilon > 0$, there exists some $\tau > 0$ such that

$$\|\bar{u}(t) - u_0\| \leq \epsilon \qquad \forall t \in [0,\tau].$$

(b) If necessary, shrink $\tau$ so that it isn't larger than the variable $\tau$ defined in Theorem 46.1, and define $I = [0,\tau]$. For the map $\mathcal{T} : C(I,W) \to C(I,W)$ defined in Theorem 46.1, show that

$$\mathcal{T}[\bar{u}](t) - \bar{u}(t) = \int_0^t e^{-A(t-s)}\Big(F(\bar{u}(s)) - F(u_0)\Big)ds.$$

(c) Show that there exists a constant $C_2$ such that

$$\Big\|\mathcal{T}[\bar{u}](t) - \bar{u}(t)\Big\| \leq tC_2 \qquad \forall t \in [0,\tau].$$

In particular, one may take $C_2 = \epsilon M K_1 e^{|a|\tau}$.

## Conclusion

The proof of Theorem 46.1 showed that $\mathcal{T} : \mathcal{F} \to \mathcal{F}$ is a contraction mapping with contraction constant $\frac{1}{2}$, hence $\lim_{n\to\infty} \mathcal{T}^n(\bar{u}) = u$, the mild solution. It then follows that:

$$\|u - \bar{u}\|_{C^0(I,W)} \leq \sum_{n=0}^{\infty} \|\mathcal{T}^{n+1}(\bar{u}) - \mathcal{T}^n(\bar{u})\| \leq \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \|\mathcal{T}(\bar{u}) - \bar{u}\| = 2C_2\tau$$

The upshot here is that our error estimate is proportional to $\epsilon K_1 \tau$. Recall that $\epsilon$ goes to zero as $\tau \to 0$ and is choosen so that $\|\bar{u}(t) - u_0\| < \epsilon$; the constant $K_1 \approx \|DF(u_0)\| + \mathcal{O}(\epsilon^2)$ bounds the Lipschitz constant of $F$; and $\tau$ is the time step. Furthermore, we never assumed that $u_0 \in \mathcal{D}(A)$; these bounds still hold if $\|Au(t)\|_W = +\infty$ for all $t$!

## #4. Global Existence and the Blowup Alternative:

Prove Lemma 47.4.