Math 876: PDE Seminar 2022
Week 9: Global Attractors

March 29 & 31, 2022

General: This week we combine all the tools we've developed up in Chapters 2, 3, and 4 to prove the existence of global attractors in PDEs. From Chapter 2, we know a global attractor exists if the semiflow on a Banach space is point dissipative and compact. From chapters 3 and 4, we've established conditions which guarantee compactness of a semiflow and local existence for nonlinear PDEs. We could even prove global existence in certain cases, but in order to show point dissipativity we need some new tricks.

Primary Reading: §4.8.1; and §5.1

errata: I think that in equation (51.12), the LHS should be $f(s)$, not $|f(s)|$.

Secondary Reading:

- §5.2 (Nonlinear Wave Equations) & §5.3 (Equations of Convection): We are primarily looking at global attractors in scalar reaction diffusion equations in §5.1 this week. But these aren't the only type of equations where global attractors exist. I encourage you to skim over §5.2 and §5.3 to see some other equations with global attractors.
- §B.2, B.3, B.4, Sobolev Inequalities, as it relates to the Nemytskii function.

Important Concepts: Energy estimates; Lyapunov function argument, Sobolev imbeddings, Nemytskii function

Reading Questions: Email me at least 3 questions on the reading at least an hour before class on Tuesday.

Presentations:

Nemytskii function (Tu): For analyzing functions $u : \Omega \to \mathbb{R}$, Hilbert spaces have nice function-analytic properties, however things get messy when we post-compose u by some polynomial $f : \mathbb{R} \to \mathbb{R}$. This process gives rise to the Nemytskii function $F[u](x) = f(u(x))$.

Discuss the Nemytskii function, and explain the proof for why polynomials $f : \mathbb{R} \to \mathbb{R}$ beget Lipschitz, Frechét differentiable functions $F : W \to W$. How do things change when we move on to dimensions $m = 2, 3$? Can you come up with an example of something that doesn't "work" in dimensions $m = 2$ or 3?

Global Attractor of 1D Chafee-Infante (Tu): Discuss the theorem and proof of Theorem 51.1, in particular focusing on the energy method and the Lypaunov function method.

Numerics for 1D Chafee-Infante (Th): Lead the discussion on problem #3 below.

Examples

For elliptic PDEs, we know that the solutions of the equation

$$\triangle u = f, \qquad\qquad u|_{\partial\Omega} = g$$

have a good deal of smoothness in the interior of Ω. By the same token, we can expect this same smoothness will be enjoyed by the equilibria of our evolutionary equations

$$0 = \partial_t u = \triangle u + F(u).$$

One place where things can get messy (in both problems) is on the boundary, in particular if the domain has a re-entrant corner. For example, consider the function

$$u(r,\theta) = r^{1/\alpha}\sin(\tfrac{\theta}{\alpha})$$

defined in the sector $0 \le \theta \le \pi\alpha$. This satisfies $\triangle u = 0$, however if $\alpha > 1$ then the derivative of u blows up as $r \to 0$. In the past I've been somewhat sloppy, writing $\mathcal{D}(A) = H^2(\Omega)$ for my Dirichlet problems when it is properly $\mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega)$. These distinction come into the spotlight this week, especially when looking at higher dimensional domains.

Problems:

1 Book Problems.

Do problems # 51.6, #51.14, and # 51.19, and # 51.21.

#2. Inhomogeneous Chafee Infante:

Do problem #51.15, which looks at the following inhomogeneous Chafee-Infante problem

$$\begin{aligned} \partial_t u - \partial_x^2 u + f(x,u) &= a\sin x & & x \in \Omega = (0,\pi), t > 0 \\ u(0,t) = u(\pi,t) &= 0 & & t > 0 \\ u(x,0) &= u_0(x), & & x \in \overline{\Omega}. \end{aligned} \tag{1}$$

where $f(x,u) = \lambda u + \beta(x)u^3$.

A look back

Recall from Weeks #1 and #2 we studied equations like (1) by first making a Fourier ansatz, that the solution could be represented as

$$u(t,x) = 2\sum_{k=1}^{\infty} a_k(t)\sin(kx) = \sum_{k\in\mathbb{Z}} -ia_k(t)e^{ikx}$$

where $a_{-k}(t) = -a_k(t)$, and the Dirichlet boundary conditions are imposed by using a sine series. If we fix parameters $a = 0$ and $\beta(x) = \beta \in \mathbb{R}$, and assume that u is a classical solution to (1), then the coefficients $a_k : \mathbb{R} \to \mathbb{R}$ satisfy the infinite system of differential equations

$$\partial_t a_k = -(\lambda + k^2) a_k + \beta \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2,k_3 \in \mathbb{Z}}} a_{k_1} a_{k_2} a_{k_3}. \tag{2}$$

Galerkin's method is to seek an approximate solution of the form $\bar{u}(t,x) = 2\sum_{k=1}^{N} a_k(t) \sin(kx)$ for some $N \in \mathbb{N}$. Plugging this ansatz into (2) we obtain a finite dimensional ODE:

$$\partial_t a_k = -(\lambda + k^2) a_k + \beta \sum_{\substack{k_1+k_2+k_3=k \\ |k_1|,|k_2|,|k_3| \leq N}} a_{k_1} a_{k_2} a_{k_3} \tag{3}$$

For example, the case $N = 2$ leads to the two-dimensional ODE

$$\begin{aligned} \partial_t a_1 &= -(\lambda + 1) a_k + \beta \left(3a_1^2 a_{-1} + 6 a_1 a_2 a_{-2} \right) \\ \partial_t a_2 &= -(\lambda + 4) a_k + \beta \left(6 a_1 a_{-1} a_2 + 3 a_2^2 a_{-2} \right) \end{aligned} \tag{4}$$

where $a_{-k} = -a_k$.

#3. Chafee Infante Numerics

(a) (Open Ended) Use numerics to explore the dynamics of the equation (3). I've posted an implementation of the exponential euler on the course Blackboard page, but you also could try writing up your own version!

(b) (Open Ended) What is the structure of the global attractor of (4)? Can you use this to make predictions about the dynamics of (3) or (1)? This low dimensional system will be a better model at some parameters, and worse at others.

(c) Consider the ODE in (3), and let $\mathfrak{A}$ denote the global attractor (assuming it exists). Show that

$$\lim_{N \to \infty} \dim \mathfrak{A} \geq \# \left\{ k \in \mathbb{N}_+ : -\lambda - k^2 > 0 \right\}.$$

(d) (Open Ended) The general philosophy behind the Galerkin/spectral approach is that as $N \to \infty$, the dynamics of the ODE and its invariant sets will converge to the dynamics and invariant sets of the PDE. In the case of solutions on the global attractor of (1), discuss why we can expect this to be the case. Where else do you think this approach will work? Where do you think it would not work?