Fractal Dimension Estimation
with Persistent Homology

J. Jaquette

Brandeis University

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2017  *On $\varepsilon$ approximations of persistence diagrams*

J. & Miroslav Kramar

2018  *Counting and Discounting Slowly Oscillating Periodic Solutions to Wright’s Equation*

J. et al.

2019  *Fractal Dimension Estimation with Persistent Homology: A Comparative Study*

J. & Benjamin Schweinhart
1 Background: Fractal Dimension and Persistent Homology

2 Previous Work and Definitions

3 Computational Results
Fractal dimension measures how the properties of a shape depend on scale.

The first notion of a fractional dimension was proposed by Hausdorff in 1918. Since then, several other definitions have been proposed, including the box-counting, packing, and correlation dimensions.

These dimensions agree on a wide class of “regular” sets.
A filtration of topological spaces is family $\{X_\alpha\}_{\alpha \in I}$ with ordered index set $I$, together with inclusions $i_{\alpha, \beta} : X_\alpha \hookrightarrow X_\beta$ for $\alpha < \beta$.

Example: if $S \subset \mathbb{R}^2$ we have the filtration of $\epsilon$-neighborhoods $\{S_\epsilon\}_{\epsilon \in \mathbb{R}^+}$. 

\[
\begin{align*}
\epsilon = 0 & \quad \epsilon = \frac{1}{32\sqrt{3}} & \quad \epsilon = \frac{1}{16\sqrt{3}} & \quad \epsilon = \frac{1}{8\sqrt{3}} & \quad \epsilon = \frac{1}{4\sqrt{3}}
\end{align*}
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Example: if \( S \subset \mathbb{R}^2 \) we have the filtration of \( \epsilon \)-neighborhoods \( \{ S_\epsilon \}_{\epsilon \in \mathbb{R}^+} \).
Persistent Homology (PH) tracks how the homology changes through a filtration. \( PH_i \) is a set of intervals corresponding to homology generators that are born and die in this process.
Definition of Persistent Homology

Given a filtration \( \{X_\alpha\} \), \( PH_i(X_\alpha) \) is the unique set of intervals so that

\[
\text{rank}(i_{\alpha,\beta} : H_i(X_\alpha) \to H_i(X_\beta)) = \# \{ I \in PH_i(X_\alpha) : [\alpha, \beta] \subseteq I \}.
\]
The information in $PH$ is often summarized by a persistence diagram: a scatter plot of (birth, death) for each interval.
$PH_1(S_\epsilon)$ is a set of intervals, one for each component of the complement that disappears as $\epsilon$ increases (by Alexander duality).
Persistent Homology

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1 hole disappears at $\epsilon = \frac{1}{4\sqrt{3}}$

1 interval $(0, \frac{1}{4\sqrt{3}})$
Persistent Homology

$PH_1(S_\epsilon)$ has one interval for each bounded component of the complement of $S$ (by Alexander duality).
Persistent Homology of a Sample

\[ \epsilon = 0 \quad \epsilon = \frac{1}{32\sqrt{3}} \quad \epsilon = \frac{1}{16\sqrt{3}} \quad \epsilon = \frac{1}{8\sqrt{3}} \quad \epsilon = \frac{1}{4\sqrt{3}} \]
If we take the persistent homology of larger and larger samples, the diagram begins to approach that of the support.
Persistent Homology of a Sample

We also have a cluster of small intervals that are usually written off as “noise.” We can use this noise to estimate fractal dimension!
**Definition (Minimum Spanning Tree)**

Let \( x \) be a finite metric space (e.g. a weighted graph). The **minimum spanning tree** on \( x \), denoted \( T(x) \) is the connected graph with vertex set \( x \) that minimizes the sum of the length of the edges.

In fact, for any \( \alpha > 0 \), \( T(x) \) minimizes the weighted sum

\[
\sum_{e \in T(x)} |e|^\alpha.
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$$\sum_{e \in T(x)} |e|^\alpha.$$
Minimum Spanning Trees and $PH_0$

If $x$ is a finite metric space, then there is a bijection between the edges of $T(x)$ and the intervals of $PH_0(x)$. An edge corresponds to an interval of half the length.

*This depends on whether persistent homology is taken with respect to the Rips/Čech complex.*
Minimum Spanning Trees and $PH_0$

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\[ PH_0(x) \]
Can the fractal dimension of a metric measure space be estimated from the persistent homology of random point samples?

How does the practical performance of the $PH_i$-dimension compare to classical methods such as box-counting or the correlation algorithm?
1. Background: Fractal Dimension and Persistent Homology

2. Previous Work and Definitions

3. Computational Results
Several authors have defined fractal dimensions based on PH, and compared computational estimates with known dimensions:


2018  Schweinhart. *PH and the Upper Box Dimension & The persistent homology of random geometric complexes on fractals:* First results relating PH to a classically defined fractal dimension.

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Persistent Homology Dimension

Definition (\(\alpha\)-Weighted Lifetime Sum)

If \(X\) is a bounded metric space, define

\[
E^i_\alpha(X) = \sum_{(b,d) \in PH_i(X)} (d - b)^\alpha.
\]

When \(i = 0\) and \(X\) is finite the sum can be taken over the edges of the minimum spanning tree on \(X\):

\[
E^0_\alpha(X) = \sum_{e \in T(X)} \left(\frac{|e|}{2}\right)^\alpha.
\]

Definition (Persistent Homology Dimension)

Let \(\mu\) be a probability measure on a metric space, \(\{x_i\}_{i \in \mathbb{N}}\) be i.i.d. samples from \(\mu\), and \(\alpha > 0\).

If the support of \(\mu\) is \(d\)-dimensional, then \(E^i_\alpha(x_1, \ldots, x_n)\) should scale as \(n^{d-\frac{\alpha}{d}}\).

\[
\dim_{PH}^\alpha(\mu) := \frac{\alpha}{1 - \beta}, \quad \beta := \limsup_{n \to \infty} \frac{\log(\mathbb{E}(E^i_\alpha(x_1, \ldots, x_n)))}{\log(n)}.
\]
Persistent Homology Dimension

**Definition (α-Weighted Lifetime Sum)**

If $X$ is a bounded metric space, define

$$E^i_\alpha (X) = \sum_{(b,d) \in PH^i(X)} (d - b)^\alpha.$$  

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$$E^0_\alpha (X) = \sum_{e \in T(X)} \left( \frac{|e|}{2} \right)^\alpha.$$

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Let $\mu$ be a probability measure on a metric space, $\{x_i\}_{i \in \mathbb{N}}$ be i.i.d. samples from $\mu$, and $\alpha > 0$.

If the support of $\mu$ is $d$-dimensional, then $E^i_\alpha (x_1, \ldots, x_n)$ should scale as $n^{\frac{d - \alpha}{d}}$.

$$\dim_{PH^i_\alpha} (\mu) := \frac{\alpha}{1 - \beta}, \quad \beta := \limsup_{n \to \infty} \frac{\log(\mathbb{E}(E^i_\alpha (x_1, \ldots, x_n)))}{\log(n)}.$$
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Ahlfors Regularity

Ahlfors regularity is a standard hypothesis that implies that the fractal dimension of a measure is well defined. That is, the various classical notions of dimension coincide and equal $d$.

**Definition (Ahlfors Regularity)**

A probability measure $\mu$ supported on a metric space $X$ is $d$-Ahlfors regular if there exist positive real numbers $c$ and $r_0$ so that

$$\frac{1}{c} r^d \leq \mu(B_r(x)) \leq c r^d$$

for all $x \in X$ and $r < r_0$.

Examples include:

- Bounded probability densities
- The natural measures on the Cantor set, Sierpinski triangle
Equivalence of Persistent Homology Dimension

Theorem (Schweinhart, 2018)

Let \( \mu \) be a \( d \)-Ahlfors regular measure on a metric space. If \( 0 < \alpha < d \) then,

\[
\dim_{PH_0}^\alpha (\mu) = d.
\]

Higher dimensional results are more difficult. Cleanest result is for \( \mathbb{R}^2 \) (for the Čech complex):

Theorem (Schweinhart, 2018)

Let \( \mu \) be a \( d \)-Ahlfors regular measure on \( \mathbb{R}^2 \) with \( d > 1.5 \). If \( 0 < \alpha < d \), then,

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Presentation Outline

1. Background: Fractal Dimension and Persistent Homology

2. Previous Work and Definitions

3. Computational Results
Question

*How does the practical performance of the PH$_i$-dimension compare to classical methods?*
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We compared the performance of algorithms to estimate the $PH_i$, box-counting, and correlation dimensions, for three classes of examples: self-similar fractals, chaotic attractors, and empirical earthquake data.
Classical Fractal Dimension Definitions

**Definition (Correlation Dimension)**

A probability measure $\mu$ on a space $X$ induces a probability measure $\nu$ on the distance set of $X$. Define the correlation integral and correlation dimension of $X$

$$C(\epsilon) = \mathbb{P}(d(x, y) < \epsilon), \quad \dim_{\text{corr}}(\mu) = \lim_{\epsilon \to 0} \frac{\log(C(\epsilon))}{\log(\epsilon)}$$

**Definition (Box Counting Dimension)**

Fix compact $X \subseteq \mathbb{R}^m$. Let $\{C^\delta_i\}_{i \in \mathbb{N}}$ be the cubes in the standard tiling of $\mathbb{R}^m$ by cubes of width $\delta$. Let $N_\delta(X)$ be the number of cubes in $\{C^\delta_i\}_{i \in \mathbb{N}}$ that intersect $X$. Define the upper and lower box-counting dimensions by

$$\dim_{\text{box}}(X) = \limsup_{\delta \to 0} \frac{\log(N_\delta(X))}{\log(1/\delta)} \quad \text{and} \quad \dim_{\text{box}}(X) = \liminf_{\delta \to 0} \frac{\log(N_\delta(X))}{\log(1/\delta)},$$
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$$\dim_{box}(X) = \limsup_{\delta \to 0} \frac{\log(N_\delta(X))}{\log(1/\delta)} \quad \text{and} \quad \dim_{box^+}(X) = \liminf_{\delta \to 0} \frac{\log(N_\delta(X))}{\log(1/\delta)}$$
Opportunities for Error

Quote (Brandstater & Swinney, 1987)

*It is not difficult to develop an algorithm that will yield numbers that can be called dimension, but it is far more difficult to be confident that those numbers truly represent the dynamics of the system.*

- Finite sampling, order of limits
- Statistical Error
- Noise
- Discretization
- Edge effects
- Lacunarity and oscillations
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\[ y = cx^{2.04} \]

**Lorenz Attractor**
Fractal dimension definitions may disagree, ... or we haven’t taken $n$ large enough, ... or we are making a systematic error.
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In general, the $PH_0$ and correlation dimensions perform comparably well. In cases where the true dimension is known, they approach it at about the same rate. In most cases, the box-counting and higher $PH_i$ dimensions perform worse.
We found one simple rule for fitting a power law to estimate the $PH_0$ which worked well for all examples, in contrast to the correlation dimension and (especially) the box-counting dimension.
We applied the dimension estimation algorithms to the Hauksson–Shearer Southern California earthquake catalog, and found a $PH_0$ dimension estimate of 1.76 and a correlation dimension estimate of 1.66. This is in line with previous studies.
For $d$-Ahlfors regular measures $\mu$, then $\dim_{PH_0^\alpha}(\mu) = d$ for all $\alpha$. We choose an $\alpha$ which gives the best convergence.
Different notions of dimension may disagree for non-regular sets. For non-regular sets, different values of $\alpha$ may give different values for $\text{dim}_{PH}^i$. 

<table>
<thead>
<tr>
<th></th>
<th>Correlation</th>
<th>Box-counting</th>
<th>$PH_0^5$</th>
<th>$PH_0^1$</th>
<th>$PH_1^5$</th>
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<th>Lyapunov</th>
</tr>
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<tr>
<td>Rulkov</td>
<td>1.01</td>
<td>1.52</td>
<td>1.62</td>
<td>1.87</td>
<td>2.02</td>
<td>&lt; 2.13</td>
<td>1.19</td>
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</table>
Slower convergence for higher dimensional data

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<tr>
<td>Lorenz</td>
<td>2.04</td>
<td>&gt; 1.90</td>
<td>2.06</td>
<td>2.05</td>
<td>&lt; 2.12</td>
<td>2.06</td>
</tr>
<tr>
<td>MG</td>
<td>3.04</td>
<td>&gt; 2.45</td>
<td>3.59</td>
<td>3.70</td>
<td>–</td>
<td>3.58</td>
</tr>
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</table>
Effectiveness.

- \(PH_0\) and correlation dimensions perform comparably well
- Box-counting, \(PH_1\), and \(PH_2\) dimensions perform worse

Efficiency.

- Computing \(PH_0\) dimension is fast and comparable with the correlation and box-counting dimensions
- \(PH_1\) dimension is reasonably fast for subsets of \(\mathbb{R}^2\)
- \(PH_1\) and \(PH_2\) are slow/impractical for higher ambient dimensions

Equivalence.

- For a large class of regular fractals \(PH\) dimension coincides with classical definitions of fractal dimension
- If the different fractal dimension estimates disagree, it may be due to (i) slow convergence (ii) systematic error (iii) definitions truly disagree

Error. Error estimates do not meaningfully reflect the difference between the dimension estimate and the true dimension

Ease-of-use. One simple rule for fitting a power law to estimate the \(PH_0\) worked well for all examples, in contrast to the correlation dimension and (especially) the box-counting dimension
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