Localized states in the generalized Swift-Hohenberg equation

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The Swift-Hohenberg equation with quadratic and cubic nonlinearities exhibits a remarkable wealth of stable spatially localized states. The presence of these states is related to a phenomenon called homoclinic snaking. Numerical computations are used to illustrate the changes in the localized solution as it grows in spatial extent and to determine the stability properties of the resulting states. The evolution of the localized states once they lose stability is illustrated using direct simulations in time.

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I. INTRODUCTION

Ever since the observation that the subcritical complex Ginzburg-Landau equation exhibits stable spatially localized states [1] there has been considerable interest in the properties of these states. The presence of these states has important consequences for other systems described by partial differential equations on the line since the Ginzburg-Landau equation describes the behavior of such systems near bifurcation from the trivial state of the system. Specifically, the complex Ginzburg-Landau equation describes the evolution of a long wavelength oscillatory instability, as well as oscillatory instabilities at finite wavelength in systems with broken reflection symmetry. In contrast, near a steady state bifurcation with finite wavelength the evolution of the instability is described by the real Ginzburg-Landau equation, and this equation possesses only unstable spatially localized states. It is of interest therefore to examine what happens to these unstable states at larger amplitude, where the real Ginzburg-Landau equation no longer provides an adequate description of the system. In this paper we show that the localized states can become stable at such amplitudes, and indeed that there is a large multiplicity of coexisting stable localized states under very general conditions. We are able to relate the existence of these states to a phenomenon sometimes called homoclinic snaking that is well known from the theory of reversible systems with 1:1 resonance, and use this theory to construct a large number of such states. The stability properties of these states are also determined, and the evolution of nonstationary localized states is studied by numerical integration in time.

It is an interesting fact that closely related phenomena have already been described in several areas involving pattern formation. The theory was originally developed in the context of water waves, where localized states have been studied by moving into a reference frame of the waves and converting the problem into an ordinary differential equation (ODE). The resulting localized states are called solitary waves, and in some cases turn out to be solitons. Kirchgrässner [2] has pioneered a successful approach to this type of problem that led to a number of advances in this area. Specifically, the ODE is viewed as a dynamical system in space, and localized states are sought as homoclinic orbits connecting the trivial state to itself. Whether such orbits are possible depends in part on the stability properties of the trivial state: eigenvalues with positive real part indicate that a nontrivial state can grow from \( x = -\infty \), while eigenvalues with negative real part indicate that such a state may return, under appropriate conditions, back to the trivial state as \( x \to \infty \). The spectrum of the linearization about the trivial state is influenced by spatial symmetries of the system. In many cases, and in particular in the case considered here, the ODE is reversible. As a result the bifurcations that are encountered as a parameter is increased are nongeneric. In the present case the spatial dynamics of the system near the trivial state turn out to be described by the reversible 1:1 resonance. The unfolding of this resonance has been worked out in detail by Iooss and Perouème [3], and can be used to understand the appearance of a variety of homoclinic orbits in this system, and hence of localized states with different spatial structure.

Parallel to these developments Pomeau [4] presented an intuitive picture of why a multiplicity of localized states might be expected. In a typical (variational) system with bistability between two spatially homogeneous states all fronts connecting such states will move in such a way that the state of lower energy density invades the state of higher energy density; stationary fronts are present at isolated parameter values only, corresponding to equal energy densities (the Maxwell point). However, this is no longer so when a front connects a homogeneous state to a spatially periodic one, since the front can “lock” to the latter. The result is the presence of a pinning region in parameter space, straddling the Maxwell point, with localized states bounded by stationary fronts at either end. Some states of this type have been computed in subsequent work [5] on the Swift-Hohenberg equation with quadratic and cubic nonlinearities, and it is this work that provides the motivation for the present paper. We show here that multiple localized states are present in three distinct regions of parameter space, and relate their origin to the presence of reversible 1:1 resonances in the spatial dynamics of the system, thereby relating Pomeau’s picture to subsequent mathematical developments, summarized in the physics literature by Coullet et al. [6], but anticipated by Champneys and colleagues in the context of localized buckling [7,8]. In fact Nishiura and Ueyama [9] identified the same type of behavior in a system of reaction-diffusion equations on the line in their study of self-replication. The latter paper has much in common with the

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work of Hilali et al. [5] in the sense that it focuses on the dynamics that result from the depinning of the Pomeau fronts, without seeking a detailed explanation of the origin of the multiplicity of localized states.

With this brief (and incomplete!) overview we introduce in the next section the model system we study and summarize its basic properties. In Sec. III we discuss the so-called single pulse localized states in this system and describe their origin and stability properties; multipulse states are not considered. The range of existence of the single pulse states is explored in Sec. IV, followed by a concluding section that considers. The range of existence of the single pulse states is explored in Sec. IV, followed by a concluding section that examines the relation of our results to earlier work by Hilali et al. and others, as well as to the mathematical theory alluded to above.

II. SWIFT-HOHENBERG EQUATION

We write the generalized Swift-Hohenberg equation in the form

$$\frac{\partial u}{\partial t} = ru - (\alpha^2 + q_c^2)u + vu^2 - gu^3$$

used by Hilali et al. [5]. Here $r$ is the control parameter while $q_c$, $v$ and $g$ are coefficients which we take to be $q_c=0.5$, $v = 0.41$, $g = 1$ as in Ref. [5]. In the presence of periodic boundary conditions with period $L$ this equation possesses a Lyapunov functional $F$ (which we refer to as an energy) given by

$$F = \int_0^L dx \left( -\frac{1}{2} ru^2 + \frac{1}{2} [(\alpha^2 + q_c^2)u]^2 - \frac{1}{3} vu^3 + \frac{1}{4} gu^4 \right)$$

such that $\frac{\delta F}{\delta u} = -\frac{\delta E}{\delta u}$. It follows that along any trajectory the energy decreases to a (local) minimum. In particular no Hopf bifurcations are possible and (at fixed $x$) all time dependence ultimately dies out.

The linear stability of a stationary solution $u_c(x)$ of period $L$ is determined by writing

$$u(x,t) = u_c(x) + \epsilon \bar{u}(x)e^{i\beta t}$$

where $\beta$ is the growth rate of the infinitesimal perturbation $\epsilon \bar{u}(x)$. Thus $\bar{u}(x)$ satisfies the eigenvalue problem

$$\beta \bar{u}(x) = \mathcal{L}[u_c(x)] \bar{u}(x), \quad \bar{u}(x + L) = \bar{u}(x),$$

where

$$\mathcal{L}[u_c(x)] = \left[ r - (\alpha^2 + q_c^2)^2 + 2vu_c(x) - 3gu_c^2(x) \right]$$

is the linear differential operator obtained by linearizing the right side of Eq. (1) about $u_c(x)$, and the growth rate $\beta$ is the associated eigenvalue.

A. Spatially homogeneous solutions

Flat, stationary solutions of Eq. (1) satisfy

$$0 = (r - q_c^2)u + vu^2 - gu^3.$$  \hspace{1cm} (6)

The three flat solution branches (shown in Fig. 1) are

![FIG. 1. The flat stationary solutions $u_0$ and $u_\pm$ as a function of $r$. Stability with respect to uniform perturbations and perturbations with wave number $q_c$, respectively, is indicated by the signs of the corresponding eigenvalues. The labeled bifurcation points are $r_0 = 0$, $r_1 = 0.020475$, $r_2 = 0.0625$, $r_3 = 0.03084$, $r_c = 0.1146$.](image)

The $u_\pm$ branches are created in a saddle-node bifurcation at $r_1 = q_c^2 - v^2/4g$. The $u_0$ branches bifurcate from the trivial solution $u_0$ in a transcritical bifurcation at $r_2 = q_c^2$.

For these states the eigenfunctions that solve Eq. (4) are $\bar{u}(x) = \sin(kx), \cos(kx)$. The corresponding growth rates on the $u_0$ and $u_\pm$ branches are

$$\beta_0 = r - (q_c^2 - k^2)^2,$$

$$\beta_\pm = 3q_c^2 - (q_c^2 - k^2)^2 - 2r - \frac{v}{2g}[v \pm \sqrt{v^2 + 8g(r - q_c^2)}],$$

respectively. For uniform states the domain size $L$ used in Eq. (4) is arbitrary so these expressions are valid for all $k = 2\pi/L$. The $u_0$ branch is unstable to uniform perturbations ($k=0$) for $r > r_2$, while the $u_\pm > 0$ branch is unstable between the saddle-node bifurcation at $r_1$ and the transcritical bifurcation at $r_2$.

Increasing $k$ up to $q_c$ results in a monotonic increase in $\beta$, so at any fixed value of $r$ the most unstable perturbations are those with wave number $k = q_c$. Increasing $k$ also widens the domain of $r$ over which the flat solutions are unstable. For each wave number $k$, the zeros of Eqs. (8a) and (8b) give the $r$ values at which the branch first loses stability with respect to perturbations of wavelength $\lambda = 2\pi/k$:

$$r_{\lambda,0} = (q_c^2 - k^2)^2,$$

$$r_{\lambda,\pm} = \frac{1}{2} \left( 3q_c^2 - (q_c^2 - k^2)^2 - \frac{v}{4g}[v \pm \sqrt{v^2 + 8g(q_c^2 - k^2)^2}] \right).$$

As a result the initial instability for all flat branches has wavelength $L_c = 2\pi/q_c$ and occurs at
Further increase of \( k \) above \( q_c \) uniformly decreases the growth rate; for values of \( k \) above \( k_{\text{max}} \), defined by
\[
k_{\text{max}} = \left( q_c^2 + \sqrt{q_c^4 + \frac{v^2}{8g}} \right)^{1/2},
\]
the growth rates \( \beta_k \) are strictly negative and so the \( u_k \) branches are stable to all short wavelength perturbations. The \( u_0 \) branch is only unstable to short wavelength instabilities at large values of \( r \).

In the following we therefore focus on solutions to Eq. (1) with \( L = L_c \), although at times it will be useful to take \( L = nL_c \), with \( n > 1 \) an integer.

**B. Primary solutions with wavelength \( L_c \) and their stability**

At the bifurcation points \( r_0 \) and \( r_4 \) branches of solutions with wavelength \( L_c \) are created. We compute the resulting branches of spatially periodic solutions using numerical continuation \cite{[10]}. The results for the branch connecting the points \( r_0 \) and \( r_4 \), hereafter referred to as the “patterned” branch and denoted by \( u_P \), are summarized in Fig. 2. Sample solution profiles are shown in Figs. 3(a) and 3(b). There are two Maxwell points of interest: at \( r_{M1} \) the patterned state has the same energy as the trivial flat state \( u_0 \), while at \( r_{M2} \) the patterned state has the same energy as \( u_c \). Another branch of \( L_c \)-periodic states, created at \( r_\infty \) on \( u_c \), extends all the way to \( r = \infty \) but is omitted from the figure. In view of the symmetry \( (v, u) = (-v, -u) \) bifurcations from \( u_c \) with \( v > 0 \) correspond to bifurcations from \( u_c \) with \( v < 0 \); such bifurcations from \( u_c \) are also analyzed in what follows.

In addition we solve the eigenvalue problem (4) for the growth rate of perturbations with spatial period \( L_c \). There is always (at least) one zero eigenvalue corresponding to the eigenfunction \( \tilde{u} = du/dx \) generating infinitesimal translations. The remaining eigenvalues are determined numerically. The results for the patterned branch are summarized in Fig. 4. This branch is unstable near both ends but gains stability through saddle-node bifurcations at \( r_3 \) and \( r_4 \). All other eigenvalues (corresponding to eigenfunctions with wavelength \( L_c / m \) for integers \( m \geq 2 \)) are negative, indicating that the patterned branch is stable with respect to short wavelength disturbances.

On larger domains the patterned states in Fig. 2(a) are subject to additional modes of instability. Figure 4(a) summarizes the resulting changes in stability while Fig. 4(b) shows the eigenvalues corresponding to eigenfunctions with wavelength \( 2L_c \). At the primary bifurcation points all the extra eigenvalues are negative. However, the eigenvalues of the two wavelength \( 2L_c \) eigenfunctions, which start at \( \beta = \beta_0 = -9q_c^4/16 \) when \( r = 0 \), change sign a total of four times, corresponding to four changes in stability of the patterned solutions with respect to wavelength \( 2L_c \) perturbations, and hence four secondary bifurcations to wavelength \( 2L_c \) solutions.
interest are small amplitude stationary states biasymptotic to $u_0$, which exist near $r=0$. The time-independent version of Eq. (1) forms a fourth order reversible dynamical system in space: the equation is invariant under the spatial reflection $(x \rightarrow -x, u \rightarrow u)$. Any localized state connecting to the trivial state $u_0=0$ as $x \rightarrow \pm \infty$ requires that $u_0$ has both stable and unstable spatial eigenvalues. It is easy to check that for $r<0$ these eigenvalues are $\pm iq \pm (\sqrt{r}/2q \pm O(r))$, while for $r>0$ they are $\pm iq \pm (\sqrt{r}/2q \pm O(r))$. Thus for $r<0$ the eigenvalues form a quartet, and $u_0$ is hyperbolic with two stable eigenvalues and two unstable eigenvalues. In contrast for $r>0$ all the eigenvalues lie on the imaginary axis and $u_0$ is not hyperbolic. In particular it is not possible to approach $u_0$ as soon as $r>0$; thus no localized states can be present when $r>0$ [12]. At $r=0$ there is a pair of imaginary eigenvalues $\pm iq$ of double multiplicity. The bifurcation at $r=0$ is thus a Hopf bifurcation in a reversible system with 1:1 resonance [3]. Theory shows that under certain conditions the hyperbolic regime ($r<0, |r|<1$) contains a large variety of spatially localized states.

Two of these states can be constructed using perturbation theory. We define the small parameter $\epsilon$ by $r=-\epsilon^2 \mu_2, \mu_2 >0$, and look for stationary solutions of Eq. (1) of the form

$$u_0(x) = \epsilon u_1(x, X) + \epsilon^2 u_2(x, X) + \cdots,$$

where $X=\epsilon x$ is a large scale over which the amplitude of the pattern changes. It follows that

$$u_1(x, X) = Z_1(X) e^{iq_{c}x} + c. c. ,$$

(13a)

$$u_2(x, X) = \frac{2v}{q_{c}^2} |Z_1|^2 + \frac{v}{q_{c}^2} Z_1^2 e^{i(q_{c}x} + Z_2(X) e^{iq_{c}x} + c. c. .$$

(13b)
The amplitude $Z(X, \epsilon) = Z_1(X) + \epsilon Z_2(X) + \cdots$ satisfies the equation

$$4q_c^2 Z_{XX} = \mu_2 Z - \gamma_1 |Z|^2 + \frac{i\epsilon}{q_c} \left( -\frac{32v^2}{27q_c} Z_X |Z|^2 + 4q_c^2 Z_{XXX} \right) + \frac{\epsilon^2}{q_c^3} \left( 8\frac{v^2}{q_c^4} Z |Z|^2 \right)_{XX} + 28Q(Z^2)_{XX} + \gamma_2 |Z|^4 + q_c^2 Z_{XXXX} + O(\mu_2^2, \epsilon^3).$$ (14)

Here

$$\gamma_3 = \frac{38u^2}{9q_c^4} - 3g, \quad \gamma_5 = -\frac{3g^2}{64q_c^4} + \frac{5795v^2}{144q^6} - \frac{3521v^4}{432q^8}.$$

Since $\gamma_3 = 8.35 > 0$ the bifurcation at the origin is subcritical. This asymptotic result is formally valid only in the limit $\epsilon \to 0$ although some of the higher order terms have been kept for later reference.

The simplest nontrivial solution of Eq. (14) is the uniform solution

$$Z(X) = (\mu_2/\gamma_3)^{1/2} e^{\phi} + O(\epsilon),$$ (15)

corresponding to spatially periodic states with period $L_c$ near $r = 0$, viz.

$$u_p(x) = 2\left( -\frac{r}{\gamma_3} \right)^{1/2} \cos(q_c x + \phi) + O(r).$$ (16)

Here $\phi$ is an arbitrary phase and $\mu_2 > 0$ (so $r < 0$). Other solutions to Eq. (14) can be found in terms of elliptic functions, and localized states correspond to infinite period solutions of this type with $Z \to 0$ as $X \to \pm \infty$:

$$Z(X) = \left( \frac{2\mu_2}{\gamma_5} \right)^{1/2} \frac{\text{sech} \left( X/\mu_2 \right)}{2q_c} e^{\phi} + O(\epsilon).$$ (17)

This solution corresponds to

$$u_i(x) = 2\left( -\frac{2r}{\gamma_3} \right)^{1/2} \frac{\sinh \left( X/\mu_2 \right)}{2q_c} \cos(q_c x + \phi) + O(r).$$ (18)

Like the spatially periodic states this family of solutions is parametrized by $\phi \in S^1$, which controls the phase of the pattern within the sech envelope. Within the asymptotics this phase remains arbitrary; there is no locking between the envelope and the underlying wave train at any finite order in $\epsilon$. However, it is known [13–15] that this is no longer the case once beyond all orders are included. These terms break the rotational invariance of the envelope solution and result in a weak flow on the circle $S^1$. This flow in turn selects specific values of the phase: $\phi = 0$ and $\phi = \pi$ [16]; since $S^1$ is a circle one of these must be weakly attracting and the other weakly repelling, cf. Fig. 10 below. At the same time these terms lead to transversal crossing of stable and unstable manifolds of $u_0$ [6] thereby producing the snaking that becomes so prominent farther away from $r = 0$. Note that the phases $\phi = 0, \pi$ are the only two phases that preserve the symmetry $(x \to -x, u \to u)$. It follows that two branches of localized states bifurcate subcritically from $r = 0$, one of which is stable with respect to translations of the envelope relative to the wave train, the other being unstable. Both states are amplitude-unstable [17]. Near $r = 0$ these two branches are essentially indistinguishable, but one might expect that with increasing $-r$ the differences will grow and therefore that two distinct branches will emerge from $r = 0$.

To prove that the localized states calculated above exist in a finite neighborhood of the origin we analyze the stationary solutions of Eq. (1) near the bifurcation at the origin using normal form theory. The appropriate normal form for the reversible Hopf bifurcation with 1:1 resonance is [3]

$$A' = iQ_1 A + B + iAP \left( \mu; |A|^2, \frac{i}{2} (AB - A^* B) \right), \quad (19a)$$

$$B' = iQ_2 B + iBP \left( \mu; |A|^2, \frac{i}{2} (AB - A^* B) \right) + AQ \left( \mu; |A|^2, \frac{i}{2} (AB - A^* B) \right), \quad (19b)$$

where in the context of spatial dynamics the prime denotes differentiation with respect to $x$, the functions $A$ and $B$ transform under spatial reflection as $(A, B) \to (A^*, -B^*)$, and $P$ and $Q$ are polynomials with real coefficients which to lowest order take the form

$$P(\mu; y, w) = p_1 \mu + p_2 y + p_3 w, \quad (20a)$$

$$Q(\mu; y, w) = -q_1 \mu + q_2 y + q_3 w + q_4 y^2. \quad (20b)$$

The bifurcation from the trivial state occurs at $\mu = 0$, and this state is hyperbolic in the region $\mu < 0$ provided $q_1 > 0$. The fact that the normal form is completely integrable [3] is of great assistance in its analysis. One finds that there are two possible types of behavior depending on the sign of $q_4$ at $q_4 = 0$. When $q_4 < 0$ homoclinic solutions are present in the whole half space $\mu < 0$ [Fig. 6(a)]. In contrast, when $q_4 > 0$ homoclinic solutions are present only for $q_4 < 0$ and then only between $\mu = 0 < \mu < 0$, where $\mu = -3q_4^2/16q_1 q_4$ [Fig. 6(b)]. At $\mu = 0$ homoclinic solutions terminate in a heteroclinic connection between the flat state $(A = 0)$ and a nontrivial state $(A \neq 0)$ with the same energy.
It should be noted that for \( \mu = -\epsilon^2 \mu_2 < 0 \), where \( \mu_2 = O(1) \), we can introduce a large spatial scale \( X = \epsilon x \). The transformation \((A, B) = (e^{A}(X), e^{B}(X))e^{\mu_2 x}\) permits us to eliminate \( B \) from the normal form, resulting in

\[
A'' = q_1 \mu_2 A + q_2 A|A|^2 + \frac{i \epsilon}{2} [-4p_1 \mu_2 A' + (6p_2 - q_3)A'|A|^2 + (2p_2 + q_3)A^3(A'' - A(A'))A' + (q_4 - q_3 p_2 + p_3^2)A|A|^4] + O(\mu^2 \epsilon^2, \epsilon^3),
\]

where the prime now denotes derivatives with respect to \( X \) and we have dropped the tildes.

A comparison of this equation with Eq. (14) permits us to identify the coefficients in the normal form in terms of the parameters of the Swift-Hohenberg equation. To transform Eq. (14) into the same form as Eq. (21) we note that it is possible to remove all second and higher order derivatives up to any order in \( \epsilon \) by iteratively replacing \( X_X \) with its power series expansion. Introducing the nonlinear transformation \( Z = A + \epsilon^2 p A|A|^2 \) allows matching at \( \mu_2 = 0 \) through order \( \epsilon^2 \), and subsequent unfolding allows matching of the \( \mu_2 \) terms up to order \( \epsilon \). The results are [18]

\[
\begin{align}
    p_1 &= -\frac{1}{8q_c^2}, & q_1 &= \frac{1}{4q_c^2}, & \rho &= -\frac{9g}{16q_c^2} + \frac{355v^2}{216q_c^2}, \\
    p_2 &= \frac{9g}{16q_c^2} - \frac{187v^2}{216q_c^2}, & q_2 &= \frac{3g}{4q_c^2} - \frac{19v^2}{18q_c^2} = -\gamma_3, \\
    p_3 &= \frac{8v^2}{9q_c^2}, & q_3 &= \frac{3g}{8q_c^2} = \frac{41v^2}{108q_c^2}, \\
    q_4 &= -\frac{177g^2}{128q_c^2} + \frac{5089gv^2}{288q_c^2} - \frac{78131v^4}{7776q_c^4}.
\end{align}
\]

Matching at higher orders in \( \epsilon \) involves higher order terms omitted from the expansion \( (20) \) of the polynomials \( P, Q \). The above results can also be obtained through a lengthier calculation of the explicit reversibility-preserving normal form transformations at \( r = 0 \), followed by appropriate unfolding.

For the parameter values used here \( q_2 < 0 \), corresponding to the subcritical case [3]. As \( q_2 \) is not small, \( q_4 \) does not play an important role in classifying the dynamics, and both Figs. 6(a) and 6(b) predict that homoclinic connections to the \( A = 0 \) state are present for \( \mu < 0 \). These are precisely the states calculated explicitly in Eq. (18). However, the scaling assumed in Eq. (12) fails for small \( q_2 \) and the localized structures then take a different form [19].

In the following we use the continuation package HomCont to extend the solutions (18) with \( \phi = 0, \pi \) to solutions of Eq. (1) valid farther away from \( r = 0 \). Technically the numerical routine finds reflection symmetric solutions to Eq. (1) on a large but finite domain \( L \), but provided the width of the resulting localized state is smaller than the domain (typically \( 40L_x \)) the true homoclinic connection is well approximated by a large period orbit. Figure 7 illustrates some of the results obtained in this manner for small \( -r \) while Fig. 8 extends these results to larger values of \( -r \). Sample profiles along each branch are shown in Fig. 9. Along the \( \phi = 0 \) branch the midpoint \( (x = 0) \) of the localized state is always a local maximum, while along the \( \phi = \pi \) branch the midpoint is always a local minimum. Near the origin the amplitude is small and the width of the sech envelope is large enough to contain many wavelengths of the underlying pattern. Away from the origin the amplitude grows and becomes comparable to the amplitude of the patterned states (specifically, the stable branch beyond the bifurcation at \( r_3 \)) and the width

![FIG. 7.](image)

![FIG. 8.](image)
Figure 8 also indicates the stability of the localized solutions in time, a consideration that is absent from the general theory of reversible systems. The eigenvalue problem (4) yields the growth rate of infinitesimal perturbations of the homoclinic solutions at each point along the branches, as well as the associated eigenfunctions \( h(x) \). The latter are localized around the base state \( u_0(x) \) and are therefore insensitive to the exact choice for \( L \). The results for the \( \phi=0 \) and \( \phi=\pi \) branches are shown in Fig. 10. This analysis confirms that both branches are unstable near the origin. The \( \phi=\pi \) branch [Fig. 10(b)] has two positive eigenvalues. Of the corresponding unstable modes one is even in \( x \) and corresponds to an amplitude perturbation of the small sech-like solution.
on the patterned branch. The termination point corresponds to a bifurcation at which a pair of homoclinic branches bi-

asymptotic to the patterned state is created, similar to the bi-

furcation at the other end of the snake at \( r = 0 \). This duality is a generic feature in problems of this type [21]. Numerical analysis beyond that presented here indicates that the loca-

tion of this bifurcation coincides with the saddle-node bifur-

cation at \( r_3 \).

A. Wavelength selection

Close examination of the wide homoclinic solutions (far up the snakes) shows that despite appearances the shape inside the envelope does not match the patterned branch per-

fectly. The largest deviation is in the wavelength, which is typi-

cally slightly larger than \( L_c \). Near the bifurcations at \( r_0 \) and \( r_n \) there exists a continuum of bifurcations to branches of solutions with wavelengths near \( L_c \). These branches look qualitatively similar to the wavelength \( L_c \) branch shown in Fig. 2. Perturbations with wavelength \( L_c \) are always the most dangerous instability for the flat states, and near such bifur-

cations the wavelength \( L_c \) branches are energetically favored over branches with other wavelengths. At large amplitude this is no longer the case, however, and on large domains we may expect the preferred solution to shift from this wave-

length. Figure 12 shows the wavelength of the preferred solu-

tion in the neighborhood of the pinning region. This curve was found by minimizing the energy density of the solutions with respect to the spatial period \( L \), at fixed \( r \). Figure 12 also shows the wavelength of the pattern within the localized states. Far up the snaking branch, where the localized state contains many wavelengths of the pattern, this wavelength is spatially uniform and independent of the width of the localized state. It is not independent of \( r \), however. Pomeau’s pinning mechanism allows for the existence of localized states away from the Maxwell point, but evidently the wave-

length of the patterned domain within such states is affected by the presence of the fronts at either side: near \( r_{p1} \), where the flat state is energetically favored, the packet is squeezed tighter, while near \( r_{p2} \), where the patterned state is favored, the packet expands. This is a frustration effect: the fronts at \( r_{p1} < r < r_{M1} \) want to move in such a way as to eliminate the localized state, leading to a compression of the state relative to its wavelength at the Maxwell point \( r = r_{M1} \) (Fig. 12). Likewise at \( r_{M1} < r < r_{p2} \) the fronts want to move outwards, thereby stretching the localized state. It is noteworthy that the resulting compression or expansion is distributed uni-

formly across the localized state, a fact that appears to be a consequence of local energy minimization. As a consequence the localized states approach a different spatially periodic state at each \( r \), although in Fig. 8 (as well as later bifurcation diagrams) we use the wavelength \( L_c \) patterned branch as a stand-in for the actual periodic state approached by the snak-

ing. Some aspects of wave-number selection at moderate am-

plitude can be captured using improved perturbation theory [19].

B. Depinning transition

The discussion summarized above indicates that below \( r_{p1} \) the (heteroclinic) fronts connecting the flat and patterned states are no longer stationary, and will instead drift in such a way that the patterned state is gradually eliminated until the lower energy flat state fills the entire domain. Above \( r_{p2} \) the fronts also cease to be stationary but now drift in such a way as to expand the patterned state until it fills the whole domain. Likewise the stationary homoclinic states that are present everywhere within the pinning region also cease to be stationary, and evolve to either the flat or patterned state. A homoclinic localized state like those shown in Fig. 8 but at a small distance outside the pinning region (for example, at \( r = r_{p2} + \delta \), with \( \delta \ll 1 \)) will evolve slowly at first because of its proximity to the saddle-node bifurcation. As it evolves farther away from the saddle node its shape will change more rapidly, its width increasing abruptly by one wavelength on either side (or decreasing, if \( r = r_{p1} - \delta \)) until it approaches the neighborhood of the next saddle-node bifurcation. Since the saddle nodes almost line up (Fig. 13) the profile will
FIG. 13. Schematic representation of the evolution of a time-dependent state just outside the pinning region.

grow in a sequence of quick “bursts” as it passes between the saddle nodes (or shrink, as it passes the ones at \( r_{P2} \)). These bursts can be viewed as nucleation events. An example of this evolution is shown in Fig. 14 in the form of a space-time plot.

The transition time \( T \) to pass between two consecutive saddle nodes follows from a semianalytic analysis of the dynamics near the bifurcation [22]. We let \( U_0(x) \) be the stationary localized state at the depinning transition \( \delta = 0 \). For \( |\delta| \ll 1 \) we then have

\[
u(x,t) = U_0(x) + \delta^{1/2} u_1(x,t) + \cdots,
\]

at least while the solution remains near \( U_0(x) \). It follows that \( u_1(x,t) \) satisfies the equation

\[
\mathcal{L}u_1 = \partial_t u_1 - |\delta|^{1/2} [\text{sgn}(\delta)U_0 + v u_1^2 - 3gU_0 u_1^2] + \mathcal{O}(\delta),
\]

where \( \mathcal{L} = \mathcal{L}[U_0] \) is the linearized Swift-Hohenberg operator (5) at \( r_{P2} \). This equation must be solved subject to the requirement that \( |u_1| \to 0 \) as \( |x| \to \infty \). Since for \( |\delta| \ll 1 \) the perturbation \( u_1 \) evolves on the timescale \( |\delta|^{1/2} \) the right hand side of this equation is uniformly small. Thus at leading order we solve \( \mathcal{L}u_1 = 0 \), i.e., the eigenvalue problem (4) restricted to the \( \beta = 0 \) subspace. At a saddle node high up each snake \( U_0(x) \) has a three-dimensional neutral eigenspace spanned by \( \tilde{U}_{10}(x) \) (the even mode whose eigenvalue vanishes at the saddle node), \( \tilde{U}_{11}(x) \) (the odd mode that tracks the even mode ever more closely as one moves up the snakes), and \( \tilde{U}_{12}(x) \) [the neutrally stable odd mode \( U_0''(x) \)]. Sample profiles of these eigenfunctions can be seen in Fig. 11 taken at one of the saddle nodes at \( r_{P2} \) located far up the snake, corresponding to a wide localized state. It follows that

\[
u(x,t) = a(t)\tilde{U}_{10}(x) + b(t)\tilde{U}_{11}(x) + c(t)\tilde{U}_{12}(x) + \mathcal{O}(|\delta|^{1/2}),
\]

where \( a(t) \), \( b(t) \), and \( c(t) \) are slowly evolving real amplitudes. The evolution equations for these amplitudes are obtained by imposing solvability conditions on Eq. (23); since \( \mathcal{L} \) is self-adjoint this condition requires that the right hand side be orthogonal to all null eigenvectors. The resulting calculation can be simplified by noting that in the space of reflection-symmetric perturbations the “center of mass” of the pattern remains fixed. Consequently we may set \( b = c = 0 \), leaving (cf. Ref. [22])

\[
a_1 \dot{a} = |\delta|^{1/2} [\alpha_2 \text{sgn}(\delta) + \alpha_3 a^2] + \mathcal{O}(|\delta|),
\]

where

\[
a_1 = \int_{-\infty}^{\infty} \tilde{U}_{10}^2 \, dx = 0.2564,
\]

\[
a_2 = \int_{-\infty}^{\infty} U_0 \tilde{U}_{10}^3 \, dx = 0.2036,
\]

\[
a_3 = \int_{-\infty}^{\infty} (v - 3gU_0) \tilde{U}_{10}^3 \, dx \approx 4.558 \times 10^{-3}.
\]

The integrals have been evaluated using numerically generated values for \( U_0(x) \) and \( \tilde{U}_{10}(x) \) at the saddle node at \( r_{P2} \). As \( \alpha_2 \) and \( \alpha_3 \) are of the same sign there are two solution branches present in \( \delta < 0 \) while for \( \delta > 0 \) the solution runs away, consistent with the location of \( r_{P2} \) on the right side of the line of saddle nodes. The transition time \( T \) to pass between successive saddle nodes is estimated as the time it takes the solution of Eq. (24) to pass from \(-\infty\) to \( +\infty \), viz.

\[
T = \frac{\pi a_1}{|\delta|^{1/2}} \approx \frac{26.44}{|\delta|^{1/2}}.
\]

Figure 15 shows the transition times, determined through explicit time integration of the type shown in Fig. 14, for several values of \( r \) near \( r_{P2} \). The predicted value (26) compares well with the best fit to the data: \( T = 24.90 |\delta|^{-1/2} \). It should be noted that an identical calculation applies near \( r_{P1} \), except that here \( \alpha_2 \) and \( \alpha_3 \) are of opposite signs.

C. Other types of localized states

The localized states of the previous section were found in a region of bistability, bounded on one side by the bifurca-
at \( r=0 \) of the patterned branch from the flat state \( u_0 \), and on the other side by the saddle node of the patterned branch at \( r_3 \), a regime examined in the context of buckling by Champneys and co-workers [7,8]. Figure 4 shows that another region of bistability is present, lying between \( r_s \) and \( r_5 \) where both the flat \( u_s \) branch and the patterned branch are stable. To study this region we shift the origin of \( r \) to \( r_s \), and define the quantities \( \bar{u} \) and \( \bar{r} \) as

\[
 u(x,t) = u_s - \bar{u}(x,t), \quad r = r_s - \bar{r}.
\]

Thus \( \bar{u} \) satisfies the equation

\[
 \frac{\partial \bar{u}}{\partial t} = \bar{r} \left( \frac{2u_{t,++}(3gu_{t,++} - v)}{q_e^4} - 1 \right) \bar{u} - (\bar{r}_s^2 + q_e^2)\bar{u} + (3gu_{t,++} - v)\bar{u}^2 - g\bar{u}^3 + O(\bar{r}^2),
\]

(27)

where \( u_{t,++} = (v + \sqrt{v^2 + 8g^2q_e^4})/4g = 0.3068 \) is the value of \( u_s \) evaluated at \( r_s \). As the \( \bar{u}^2 \) coefficient is positive, Eq. (27) is of the same form as the original Eq. (1), and the behavior near \( r_s \) in the new bistable region is mathematically analogous to that described in the preceding section, although this possibility is absent from Refs. [7,8] owing to a different parametrization. In particular, the asymptotic analysis and the computation of the normal form coefficients carried out in Eqs. (12)–(18) and (22) carries over to this case using the transformation

\[
 u \rightarrow u_s - u, \quad r \rightarrow (r_s - r) \left( \frac{2u_{t,++}(3gu_{t,++} - v)}{q_e^4} - 1 \right), \quad v \rightarrow 3gu_{t,++} - v.
\]

(28a, 28b, 28c)

For example, at \( r = r_s \), it follows that this region also contains two branches of homoclinic solutions which, like the patterned branch, bifurcate from the \( u_s \) branch at \( r_s \). As shown in Fig. 16 these homoclinic branches undergo a series of saddle-node bifurcations between \( r_{P1} \) and \( r_{P4} \). As in the previous case, there is a Maxwell point \( r_{M2} \) within the pinning region where the energy of the flat \( u_s \) branch matches that of the patterned branch. Below this point the patterned state has the lower energy density while above the \( u_s \) state is preferred. However, once again the patterned branch \( u_p \) shown in Fig. 16 has a wavelength that does not exactly match the pattern inside the localized states that make up the snakes: near \( r_5 \) and \( r_{P4} \) the wavelength within the localized states deviates by as much as 15% from \( L_s \). The wavelength discrepancy is most noticeable at large \( r \) where the flat state has lower energy density, squeezing the packets so that the observed wavelength within the localized states is less than \( L_s \). It is significant that the patterned branch with this wavelength does not undergo a period-doubling bifurcation at \( r_5 \). As a result the localized states on the snakes do not inherit the change of stability of \( u_p \) at \( r_5 \) or indeed any of the bifurcations to \( nL_s \)-periodic states that are present near \( r_5 \), and remain stable all the way up to the saddle node bifurcations at \( r_{P4} \).

As mentioned earlier bifurcations from the \( u_+ \) branch can be mapped to the \( u_+ \) branch. As a result the bistable region that exists between \( u_- \) and the patterned branch can be studied within the same framework. The remaining possibility, branches of solutions connecting the \( u_- \) and \( u_+ \) states, can be excluded on energy grounds: if \( v > 0 \) (\( v < 0 \)) the energy of
the $u_*$ state is always less than (greater than) the $u_-$ state. Moreover, no pinning is possible. Thus no stationary fronts between these two flat states are possible when $v \neq 0$. Kink solutions, present when $v = 0$, are discussed in Ref. [8].

IV. LOCALIZED STATES AS A FUNCTION OF $v$

Thus far we have studied the behavior of Eq. (1) as $r$ varies at fixed values of $q_c$, $v$, and $g$. To explore the $v$ dependence of our results we rescale $x$ and $u$ so that $q_c = 0.5$ and $g = 1.0$, and vary $v$. Consider first the bifurcation from the $u = 0$ flat state, shown in Fig. 8 at $v = 0.41$. Within the normal form (21) changing $v$ corresponds to changing $q_2$. The condition $q_2 = 0$ defines a codimension-2 point in the $(r, v)$ plane with coordinates $r = 0$, $v = \pm \sqrt{27gq_2^4/38}$; without loss of generality we take $v > 0$. At this point $q_4 = 2202g_2^2/361q_2^6 > 0$ and the reversible Hopf bifurcation at $r = 0$ in Eq. (1) is of the type shown in Fig. 6(b). In this case normal form theory predicts that homoclinic solutions only exist in the subcritical regime $q_2 < 0$ (i.e., $v^2 > 27gq_2^4/38$) and only in the range $\mu_0 < \mu < 0$. The heteroclinic solution at $\mu_0$ in the normal form corresponds to the Maxwell point $r_{M1}$ in the partial differential equation. Away from the codimension-2 point we can use the normal form to confirm the existence of homoclinic solutions when $r \leq 0$ and it is these small amplitude solutions that provide a starting point for the numerical continuation to large amplitude used to determine the extent of the pinning region. We can also follow the Maxwell point.

Figure 18 summarizes the regions of existence of heteroclinic connections to the trivial flat state $u_0$. In view of the parameter symmetry $(u, v) \to (-u, -v)$ of Eq. (1) a mirror image of this picture exists in the $v < 0$ region of parameter space. The bifurcation diagram in Fig. 8 corresponds to a horizontal slice through this figure at $v = 0.41$ and is typical of the behavior below $v = 0.688$. Above this value of $v$ a new Maxwell point, corresponding to equal energies of the $u_0$ and $u_+$ states,

$$r_{M3} = q_4^2 - \frac{2v^2}{9g},$$

becomes dynamically important. For $v > \sqrt{9gq_4^2/2}$ this Maxwell point lies to the right of the bifurcation at $r_+$, within the region where $u_+$ is stable, and at $v = 0.688$ $r_{M3} = r_{P2}$. Thus for $v \approx 0.688$ (i.e., above the horizontal boundary of the shaded region in Fig. 18) the new Maxwell point enters the pinning region around $r_{M1}$ and the structure of the flat and patterned states changes, as do the homoclinic branches. In particular, above $v = 0.735$ the $\phi = 0, \pi$ homoclinic branches created at the origin undergo homoclinic snaking towards the $u_-$ state instead of $u_+$. Since $u_+$ is a spatially homogeneous state no pinning occurs, and the snakes collapse asymptotically to a single point at $r = r_{M1}$. Thus at $r_{M3}$ an infinite number of homoclinic states of different lengths bi-symmetric to $u_0$ is still present, but away from $r_{M3}$ only a finite number of such states remains [23]. Bifurcation diagrams describing these homoclinic states for $\phi = 0, \pi$ are shown in Fig. 19. The solutions on the $\phi = \pi$ branch, shown in Fig. 19(b), include a small region of width of order $L_c$ where the solution profile dips back down to $u = 0$ in order that $x = 0$ remains a minimum.

Between $v \sim 0.688$ and $v \sim 0.735$ there is a transition region as the homoclinic snaking in the pinning region around $r_{M1}$ (Fig. 8) shifts to straddle the new heteroclinic connection at $r_{M3}$ (Fig. 19). This intermediate region (Fig. 20) is complicated by the existence of yet another new Maxwell point, labeled $r_{M4}$, between $u_0$ and the unstable section of the patterned branch near $r_*$. This Maxwell point is close to $r_{M3}$ because near $r_*$ the energies of the flat and patterned states are very similar. The profiles along both the $\phi = 0$ and $\phi = \pi$ branches contain domains within which $u(x)$ resembles three different states: $u_0$, the (unstable) solution from the upper segment of the patterned branch, and the (stable) solution from the middle segment. As one proceeds up each branch the fronts between the flat and unstable pattern move apart filling most of the domain with the unstable pattern. However, the fronts between the two structured states remain fixed leaving a small patch of the stable pattern near the origin. It is this “double” structure that is responsible for the complex structure of the corresponding snakes, cf. Ref. [24].
Note in particular the unusual looping on the lower part of the $\phi=0$ branch (absent from the $\phi=\pi$ branch) whose origin remains unclear. An analysis of the eigenvalues shows that all such states far up the snaking branches are unstable.

The snaking shown in Fig. 20 is a consequence of the Maxwell point $r_{M1}$ but two other Maxwell points are present as well. Snaking about the latter is illustrated in Figs. 16 and 19. It is natural to ask what selects the Maxwell point about which a snake develops, and if snakes abandon one Maxwell point in favor of another as parameters are varied how such transitions take place. For example, Fig. 18 suggests an abrupt end to the pinning region surrounding $r_{M1}$ at $v=0.688$ that may mark a genuine limit point above which the pinning region abruptly end to the pinning region surrounding $r_{M1}$ at $v=0.688$ that may mark a genuine limit point above which the pinning region.

FIG. 19. (Color online) Bifurcation diagram corresponding to a horizontal slice through Fig. 18 at $v=0.75$ illustrating (a) $\phi=0$ and (b) $\phi=\pi$ homoclinic snaking around a heteroclinic connection between the two flat states $u_0$ and $u_\lambda$. Thick lines indicate stable solutions; asymmetric states are omitted. The insets show typical solution profiles as the envelope widens and fills with the $u_\lambda$ state.

The snaking shown in Fig. 20 is a consequence of the Maxwell point $r_{M1}$ but two other Maxwell points are present as well. Snaking about the latter is illustrated in Figs. 16 and 19. It is natural to ask what selects the Maxwell point about which a snake develops, and if snakes abandon one Maxwell point in favor of another as parameters are varied how such transitions take place. For example, Fig. 18 suggests an abrupt end to the pinning region surrounding $r_{M1}$ at $v=0.688$ that may mark a genuine limit point above which the snaking branch transitions to one of the other Maxwell points. One may think that only a single Maxwell point can be graced by snakes since only a single pair of localized states ever bifurcates from the flat state. Whether snakes can split at finite amplitude into multiple pairs remains unclear. Moreover, as discussed further below, snaking does not even require the presence of Maxwell points.

We next consider homoclinic states biasymptotic to $u_\lambda$. It is easy to show using Eq. (28) that at $r_\lambda$ the coefficient $q_0$ <0 for all values of $v$. Hence this bifurcation is always subcritical and no codimension-2 point occurs from which heteroclinic states can be traced. Small amplitude homoclinic solutions near $r_\lambda$ remain, however, and are given by Eqs. (18) and (28). These can be followed numerically to larger amplitudes. Figure 21 shows the location in $(r,v)$ phase space of the pinning region that contains solutions heteroclinic to $u_\lambda$. At each point within this region there exists an infinite number of homoclinic solutions that differ only in width. We identify three possible structures that describe the organization of these states along one-parameter slices through this figure. The first is regular homoclinic snaking, shown previously in Fig. 16, with all states joined into a finite number of branches according to the phase $\phi$ (here $\phi=0,\pi$). The second possibility may be termed semi-infinite snaking and arises when one of the lines of saddle nodes moves off to infinity. The third case, which we call homoclinic loops, consists of an infinite stack of disconnected isolas, each made up of solutions with a fixed number of oscillations; this number increases from isola to isola as one moves up the stack.

Close inspection of Fig. 22 shows that both Figs. 4 and 16 are incomplete. In each case the complete picture resembles that shown in Fig. 22 but includes an extra stack of disconnected patterned states that extend to arbitrarily large $r$ and form what we have called a semi-infinite snake. It is interesting that the latter is not associated with a Maxwell point: the only Maxwell point between the flat and patterned states occurs at $r_{M2}$ in the regular homoclinic snaking region. Both the regular and semi-infinite snaking regions apparently persist to arbitrarily large $v$. 

FIG. 20. (Color online) Bifurcation diagram corresponding to a horizontal slice through Fig. 18 at $v=0.70$. In (a) the localized states correspond to $\phi=0$ while (b) shows the $\phi=\pi$ branch. Thick lines indicate stable solutions; asymmetric states are omitted. The insets show typical solution profiles far up the snaking branches. The larger (smaller) amplitude states correspond to the stable middle (unstable upper) segments of the $u_P$ branch.
extends beyond the patterned branches. This is a consequence of the wavelength selection mentioned above. The patterned branch shown in the figure has wavelength $\lambda_{\bar{u}}$; at large $r$ this wavelength differs from that within the localized state. Inspection of the wave number of the pattern within these states shows that the preferred value is about 30% higher. The patterned branch corresponding to this wave number extends all the way back to the pinning transition, providing support for all the localized states in the semi-infinite snaking region.

As $v$ decreases below $v = 0.3733$ the states in Fig. 22 merge in a “zipping” transition [25], leaving a single semi-infinite snake that extends from $r_{P3}$ to $r = \infty$. It is remarkable that this region includes the case $v = 0$ corresponding to the standard Swift-Hohenberg equation with no quadratic nonlinearity. The corresponding bifurcation diagram is shown in Fig. 23 and is typical of this region. The case $v = 0$ has been studied in the past (e.g., Refs. [22,26]) but the disconnected branches of (stable) localized states have not been noted before. Indeed, in view of the general scaling $r \sim q_c^3$ we find that $r_{P3} = 0.10943 \approx 1.756q_c^3$ at $v = 0$, a prediction that agrees with the location of the pinning transition found in Ref. [22] ($r = 1.74$, $q_c = 1$). Likewise the Maxwell point occurs at $r_{M2} = 0.355 \approx 5.68q_c^3$, as found in Ref. [27].

Below $v \approx -0.03$ the semi-infinite snake “unzips” into two segments in $r$, one finite and one extending to $r = \infty$. In the finite segment the bifurcation diagrams (Fig. 24) are now topologically distinct from those in Fig. 22, consisting of stacks of homoclinic isolas instead of snakes. Each isola contains a segment with stable localized states. At these values of $v$ the two branches of patterned solutions created at $r_0$ and $r_+$ extend to infinite $r$, and the semi-infinite snaking region does contain a Maxwell point where the flat and patterned states have equal energies. However, the isola region possesses no such point. Further decrease below $v = -0.109$ causes the homoclinic isolas to shrink and vanish, but the semi-infinite snaking region remains at arbitrarily large negative $v$.

At first glance the two different regions of homoclinic snaking identified in the previous section (one created at $r_0$ and shown in Fig. 8 and the other at $r_+$ and shown in Fig. 16) appear similar but despite the mathematical similarity between the bifurcations at $r_0$ and $r_+$ this is not the case. The saddle node at $r_3$ is created when stable and unstable branches of patterned solutions pinch off and disconnect, in contrast to what happens at $r_5$. Thus the semi-infinite snakes are likely inherited from snakes that are ultimately associated with Maxwell points at smaller values of $r$. There is no ana-
theory of this bifurcation provides a key to understanding the origin of the localized states. This theory requires the use of beyond-all-orders asymptotics [13,14,28]; for this reason the phase selection between the \( \phi=0 \) and \( \phi=\pi \) branches is initially extremely weak, as is the amplitude of the snake. Both processes only become significant in the pinning region. High up each snake the localized states become broad and begin to resemble heteroclinic orbits, i.e., slugs of a periodic wavetrain embedded in a flat background.

We have seen that the generalized Swift-Hohenberg equation admits multiple Maxwell points, and hence multiple pinning regions exhibiting snapping. We have focused on three such regions. The first, present for \( r<0 \), occurs when the trivial and patterned states have equal energies. The second and third arise when the patterned state has the same energy as the nontrivial flat states \( u_{\pm} \). The corresponding pinning regions describe the location of localized states that are bias asymptotic to a nonzero flat state. In other regimes we have located infinite stacks of isolas of localized states. However, perhaps the most unexpected observation is that there is a broad range of values of \( \nu \) for which stacks of localized states come in from infinity (in \( r \)), turn around in a saddle-node bifurcation, and return to infinity. These states are highly nonlinear, acquire stability at the saddle-node bifurcations, and are present even when \( \nu = 0 \), i.e., in the usual Swift-Hohenberg equation with only a single stabilizing cubic nonlinearity. Throughout the paper we have only considered the so-called single pulse solutions. Multipulse solutions, consisting of several more or less isolated localized states interacting via their tails, possess rich behavior in their own right [14,21,28], but are beyond the scope of the present work.

Localized states are of course of great interest to pattern formation. They occur not only in vibrating granular media [29] and polymeric fluids [30], but also in reaction-diffusion systems [9], nonlinear optics [31–33], ferrofluids in a magnetic field [34], neuronal networks [35], and in several convection systems [36–42]. They are also present in the Swift-Hohenberg equation with cubic and quintic nonlinearities [20,43]. Two-dimensional localized patterns also exhibit snapping [6,33,44]. In particular, the theory of localized buckling of long struts bears a substantial similarity to the work reported above [7,8]. This in turn is closely related to the study of solitary waves in the fifth order Korteweg–de Vries (KdV) equation arising in the theory of long wavelength water waves [45,46] and related systems [47]. In particular the solitary waves studied in the context of the fifth order KdV equation correspond precisely to the localized states of the Swift-Hohenberg equation (1) in the special case \( g=0 \), with the parameter \( r \) related to the speed of the waves. Indeed, it is in this context that the bifurcation of the \( \phi=0, \pi \) branches of localized states from the trivial state was first established [48], the dominant nonlinearity near \( r=0 \) being quadratic. Moreover, no other branches of localized states bifurcate from \( r=0 \) [14]. It is worth mentioning that the water wave problem, like the buckling problem, is fundamentally time independent. Thus the question of linearized stability does not arise in a natural way, unless additional physical notions such as wave radiation to infinity are included. It is for this reason that stability questions have not been central to the
study of localized states in great many cases, although in other contexts the stability of the solutions along the snakes has been studied [33,49] albeit with respect to symmetric perturbations only.

It is worth mentioning that in the large body of work on localized buckling the load parameter enters as the coefficient of the second derivative. Thus in this class of problems the parameter that is varied is the wave number \( q_z \), while keeping a particular combination of \( r \) and \( q_z^4 \) constant. This procedure represents a distinct cut through the parameter plane that encounters only one of the possible Maxwell points, corresponding to our first snaking region (near \( r=0 \)). We surmise that this is the reason why the interesting behavior in and near the other snaking regions described here has not already been noted.

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[12] More precisely, no localized states with exponential approach to \( u_0=0 \) are possible when \( r>0 \).
[16] More generally the terms beyond all orders select an even number of fixed points, but the additional localized states cannot bifurcate from \( r=0 \).