COMPUTATIONAL TOOLS FOR QUADRATIC CHABAUTY

JENNIFER S. BALAKRISHNAN AND J. STEFFEN MÜLLER

Contents

Date: April 11, 2023.

1. Introduction

The *quadratic Chabauty* method is the first nonabelian step of Kim's program for achieving an algorithmic determination of the set $X(\mathbb{Q})$ of rational points on a nice[1] curve $X/\mathbb{Q}$ of genus g of 2 or more. The quadratic Chabauty set $X(\mathbb{Q}_p)_2 \supset X(\mathbb{Q})$ is a finite subset of $X(\mathbb{Q}_p)$ for those curves with good reduction at p and Jacobian J having Mordell–Weil rank r and Néron–Severi rank $\rho(J)$ satisfying the hypothesis

$$r < g + \rho(J) - 1.$$

In these notes[2], we develop tools for carrying out the quadratic Chabauty method in the case when $r = g$ and $\rho(J) \geq 2$, with a focus on algorithmic and computational[3] aspects. The goal of these notes are two-fold: first, to serve as a user's guide for those interested in getting started with the quadratic Chabauty method, and second, to highlight some interesting problems along the way.

Kim's nonabelian Chabauty program is a vast generalization of the Chabauty–Coleman method. The latter solely uses *abelian* geometric data: the structure of the Jacobian, as well as p-adic abelian integrals. It applies to curves satisfying the hypothesis $r < g$ and relies on the construction of an annihilating differential, which essentially can be computed using p-adic linear algebra.

Since the classical Chabauty–Coleman method motivates some of our framing of the quadratic Chabauty method, we begin our discussion by giving a survey of the tools used to carry out the former method, where there are still a number of tractable computational challenges. The main construction here is how p-adic (Coleman) integrals can be computed using p-adic cohomology. Then when the Chabauty–Coleman hypothesis is satisfied, one can use the calculation of Coleman integrals to compute a finite set of points $X(\mathbb{Q}_p)_1 \supset X(\mathbb{Q})$.

We also describe how n-fold iterated Coleman integrals can be computed, which in the case of $n = 2$, provides input into computations involving p-adic heights. We then survey a few constructions of p-adic heights in various settings, which leads into the quadratic Chabauty method. We briefly describe how this fits into Kim's nonabelian Chabauty program, though a more comprehensive treatment of the theory will be covered in Kim's lecture course. Finally, we combine the algorithms for quadratic Chabauty to carry out an example to determine rational points on the Atkin–Lehner quotient modular curve $X_0^+(167)$, which has genus 2 and rank 2.

Throughout, we illustrate our techniques with examples, and where possible, we include or link to code snippets for carrying out computations in `SageMath` [The20] or `Magma` [BCP97].

[1] Throughout, by a *nice* curve, we mean one that is smooth, projective, and geometrically irreducible.

[2] These are lecture notes for the course "Computational tools for quadratic Chabauty", taught by JB at the 2020 Arizona Winter School on Nonabelian Chabauty. They were originally planned as a combined set of lecture notes for this course and an additional course, "Quadratic Chabauty", taught by SM at the 2020 AWS. SM withdrew his participation after realizing that, in contrast to previous editions, the 2020 edition of the school would be supported by the NSA.

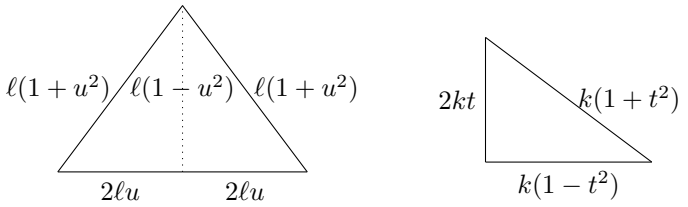
[3] While computations of p-adic objects are usually not exact, one can analyze the precision necessary to produce provably correct results.

We note two further perspectives on quadratic Chabauty: *geometric quadratic Chabauty*, pioneered by Edixhoven–Lido [EL23], with lecture notes by Lido on Edixhoven's 2020 AWS course elsewhere in this volume. Geometric quadratic Chabauty uses line bundles over the Jacobian, the Poincaré torsor and models over the integers. More recently, Besser–Müller–Srinivasan [BMS21] describe *p-adic Arakelov quadratic Chabauty*, which gives a new construction of p-adic heights on varieties over number fields using p-adic adelic metrics on line bundles, in the spirit of Zhang's construction of real-valued heights via adelic metrics [Zha95].

1.1. **A question about triangles.** We start with a question from Euclidean geometry that leads to an interesting Diophantine problem. We say that a *rational triangle* is one all of whose side lengths are rational.

Question. *Do there exist a rational right triangle and a rational isosceles triangle that have the same area and the same perimeter?*

This would mean that we have a pair of triangles with the following side lengths:



Let us rescale so that we may assume $\ell = 1$. We further suppose that $k, t, u \in \mathbb{Q}$, $0 < t, u < 1$ and $k > 0$. By equating areas and perimeters, we obtain the following system of equations:

$$k^2 t(1-t^2) = 2u(1-u^2)$$
$$k + kt = 1 + 2u + u^2$$

Let $x = 1 + u$. After some algebra, we see that there is $x \in \mathbb{Q} \cap (1, 2)$ such that

$$2xk^2 + (-3x^3 - 2x^2 + 6x - 4)k + x^5 = 0.$$

Then noting that the discriminant of the polynomial in k is a rational square, we have that

$$y^2 = (-3x^3 - 2x^2 + 6x - 4)^2 - 4(2x)x^5,$$

and simplifying, this gives us a genus 2 curve

$$X : y^2 = x^6 + 12x^5 - 32x^4 + 52x^2 - 48x + 16.$$

We would like to compute the set of rational points $X(\mathbb{Q})$ on X. Some useful input now is knowing the Mordell–Weil rank of the Jacobian of X: it turns out that the rank is equal to 1. In general, computing the rank of a Jacobian is a difficult problem, but `Magma` has an implementation of 2-descent on Jacobians of hyperelliptic curves that can be used here:

```
> R<x>:=PolynomialRing(RationalField());
> X:=HyperellipticCurve(x^6+12*x^5-32*x^4+52*x^2-48*x+16);
> J:=Jacobian(X);
> RankBounds(J);
1 1
```

The output of `RankBounds` is a lower bound on rank, followed by an upper bound on rank, which are both equal to 1. Consequently, the *Chabauty–Coleman* bound (more on this in a bit; see Theorem 1.4 if you'd like to skip ahead) gives

$$\#X(\mathbb{Q}) \leq 10.$$

Are there rational points on X? After searching in a box, we find

$$\{\infty^{\pm}, (0, \pm 4), (1, \pm 1), (2, \pm 8), (12/11, \pm 868/11^3)\} \subseteq X(\mathbb{Q}),$$

and we have found precisely 10 points. So we have determined $X(\mathbb{Q})$, and the rational point $(12/11, 868/11^3)$ gives us a unique pair of triangles.

Theorem 1.1 (Hirakawa–Matsumura [HM19]). *Up to similitude, there exists a unique pair of a rational right triangle and a rational isosceles triangle which have the same perimeter and the same area. The unique pair consists of the right triangle with side* $(377, 135, 352)$ *and isosceles triangle with sides* $(366, 366, 132)$.

We begin with some context for these results. It was conjectured by Mordell in 1922 that nice curves of genus 2 or more have only finitely many rational points. This was proved by Faltings:

Theorem 1.2 (Faltings [Fal83]). *Let* $X/\mathbb{Q}$ *be a nice curve of genus* ≥ 2. *Then the set* $X(\mathbb{Q})$ *is finite.*

How do we determine the set $X(\mathbb{Q})$? Faltings' proof is not constructive. There is another proof due to Vojta [Voj91] (also revisited by Bombieri [Bom90]), but it is also not effective. We note that the recent proof of Mordell's conjecture by Lawrence and Venkatesh [LV20] gives another approach to finiteness, see also [BBB+21].

One method that allows us to compute the set $X(\mathbb{Q})$ in some cases is known as the Chabauty–Coleman method (see [MP12] for a beautiful introduction to the circle of ideas involved, as well as [Sto] for an excellent overview of the relevant techniques and further developments in the case of hyperelliptic curves). This is due to Coleman [Col85b], who re-interpreted earlier work of Chabauty [Cha41] in proving Mordell's conjecture in the following special case:

Theorem 1.3 (Chabauty, 1941). *Let* $X/\mathbb{Q}$ *be a nice curve of genus* $g \geq 2$. *Suppose the Mordell–Weil group of* J *has rank* $r < g$. *Then* $X(\mathbb{Q})$ *is finite.*

Coleman gave an effective version of Chabauty's theorem:

Theorem 1.4 (Coleman [Col85a]). *Let* $X/\mathbb{Q}$ *be a nice curve of genus at least* 2. *Suppose the Mordell–Weil rank of* $J(\mathbb{Q})$ *is less than* g. *If* $p > 2g$ *is a prime of good reduction for* X,

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$

This result comes from bounding the number of zeros of a p-adic (Coleman) integral. We will say a bit more about this later.

Going back to the triangle problem: recall that we have the genus 2 curve

$$X : y^2 = x^6 + 12x^5 - 32x^4 + 52x^2 - 48x + 16.$$

The curve X has good reduction at $p = 5$, and we compute the set of $\mathbb{F}_5$-rational points:

$$X(\mathbb{F}_5) = \{\infty^{\pm}, (0, \pm 1), (1, \pm 1), (2, \pm 2)\},$$

so $\#X(\mathbb{F}_5) = 8$. Thus by Coleman's theorem, we have

$$\#X(\mathbb{Q}) \leq 8 + 2 \cdot 2 - 2 = 10.$$

Since the Chabauty–Coleman method involves p-adic integration of certain differentials, we first set some notation on differentials and then discuss p-adic integration. We assume throughout that p is a prime of good reduction for a nice curve X.

Definition 1.5. Let X be a nice curve over a field k. The set of (meromorphic) differentials on X over k forms a 1-dimensional $k(X)$-vector space $\Omega^1(k)$.

Definition 1.6. Let $0 \neq \omega \in \Omega^1(k)$ and $P \in X(k)$. Let $t \in k(X)$ be a uniformizer at P, and use this to write $\omega = \omega(t)dt$. Then $v_P(\omega) := v_P(\omega(t))$ is the *valuation of ω at P*.

Definition 1.7. If $v_P(\omega) \geq 0$ (or $\omega = 0$), then we say that ω is *regular at P*. We say that ω is *regular* if it is regular at all points $P \in X(\bar{k})$. This is also known as a differential of the *first kind*. A differential of the *second kind* is a differential that has residue zero at all points $P \in X(\bar{k})$. A differential of the *third kind* has at most simple poles at all points.

Example 1.8. Let $X\colon y^2 = f(x)$ be a hyperelliptic curve of genus g over k. Then $H^0(X, \Omega^1)$ has basis

$$\left\{\frac{dx}{2y}, \frac{xdx}{2y}, ..., \frac{x^{g-1}dx}{2y}\right\}$$

so every regular differential can be uniquely written as $\frac{p(x)dx}{2y}$, with a polynomial p of degree $\deg(p) \leq g-1$.

Now we begin with an introduction to Coleman's theory of p-adic line integration. We start with a list of the relevant properties of this integral when the integrand is a regular differential.

Theorem 1.9 (Coleman [Col82, Col85b])**.** *Let $X/\mathbb{Q}_p$ be a nice curve with good reduction at p. For each pair of points $P, Q \in X(\overline{\mathbb{Q}}_p)$ and regular differential $\omega \in \mathrm{H}^0\left(X, \Omega^1\right)$ we can define a (p-adic) Coleman integral*

$$\int_P^Q \omega \in \overline{\mathbb{Q}}_p,$$

which satisfies the following properties:

1. *Linearity:*
$$\int_P^Q (a\omega + b\eta) = a\int_P^Q \omega + b\int_P^Q \eta.$$
2. *Additivity in endpoints:*
$$\int_P^Q \omega = \int_P^R \omega + \int_R^Q \omega.$$
3. *The integral from a point P to itself satisfies*
$$\int_P^P \omega = 0.$$
4. *For a degree-zero divisor $D = \sum_{j=1}^n ((Q_j) - (P_j))$ on X, the integral*
$$\int_D \omega := \sum_{j=1}^n \int_{Q_j}^{P_j} \omega$$
is well-defined.
5. *If D is a principal divisor, then $\int_D \omega = 0$.*
6. *Galois compatibility: If K is a finite extension of $\mathbb{Q}_p$ such that $P, Q \in V(K)$ ω is defined over K, then $\int_P^Q \omega \in K$.*

(7) *Fix $P_0 \in X(\overline{\mathbb{Q}}_p)$ and $\bar{P} \in X(\overline{\mathbb{F}}_p)$ and let $\omega \neq 0$ be a regular differential on X. Then there are only finitely many $P \in X(\overline{\mathbb{Q}}_p)$ such that $\int_{P_0}^{P} \omega = 0$ and $P \equiv \bar{P} \bmod p$.*

Remark 1.10. If P and Q reduce to the same point $\bar{P} \in X_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)$, then we call the Coleman integral a *tiny integral*. It can be evaluated in the following intuitive way: Expand ω into a power series $\omega(t)dt$ in a uniformizer t at P that reduces to a uniformizer at $\bar{P}$. Let ℓ be the power series ℓ such that $d\ell(t) = \omega(t)dt$ and $\ell(0) = 0$; then $\int_P^Q \omega = \ell(t(Q))$. This is independent of the choice of t.

Remark 1.11. One can extend the theory of Coleman integrals to bad reduction. In fact, there are a number of closely related approaches to p-adic integration, by Berkovich [Ber07], Zarhin [Zar96], Colmez [Col98], Besser [Bes02], and Vologodsky [Vol03]. See also the excellent survey of Besser [Bes12]. We also refer to Zureick-Brown's lectures for more on (single) p-adic integrals.

Properties (4) and (5) of the Coleman integral allow us to extend it to the Jacobian. The following result is then immediate.

Corollary 1.12. *Given the hypothesis of the previous theorem, assume that there is a point b in $X(\mathbb{Q}_p)$, let J be the Jacobian of X, and let*

$$\begin{aligned} i\colon X &\to J \\ P &\mapsto [(P) - (b)] \end{aligned}$$

be the Abel-Jacobi embedding of X into J. Then there is a map

$$\begin{aligned} J(\mathbb{Q}_p) \times H^0\left(X_{\mathbb{Q}_p}, \Omega^1\right) &\to \mathbb{Q}_p \\ (Q, \omega) &\mapsto \langle Q, \omega \rangle \end{aligned}$$

that is additive in Q and $\mathbb{Q}_p$-linear in ω and is given by

$$\langle [D], \omega \rangle = \int_D \omega$$

for $D \in \mathrm{Div}_X^0(\overline{\mathbb{Q}}_p)$. In particular, for $P \in X(\mathbb{Q}_p)$, we have the Abel-Jacobi morphism AJ_b that takes P to the linear functional

$$\langle i(P), \omega \rangle = \int_b^P \omega =: \mathrm{AJ}_b(P).$$

Remark 1.13. Since our focus is computational, we do not discuss the construction of the Coleman integral. See [MP12] for one way to do it: One first defines the abelian integral on the Jacobian using the structure of $J(\mathbb{Q}_p)$ as a p-adic abelian Lie group and pulls this back to the curve. With this approach, one starts with Corollary 1.12 and deduces properties of the integral on the curve from it.

Remark 1.14. A torsion point $P \in J(\mathbb{Q}_p)$ satisfies $\langle P, \omega \rangle = 0$ for all $\omega \in H^0(X_{\mathbb{Q}_p}, \Omega^1)$. To see this, if $nP = 0$, then $\langle P, \omega \rangle = \frac{1}{n}\langle nP, \omega \rangle = 0$. One can show that no other points have this property.

In the Chabauty–Coleman method, we will make use of a certain subspace of the space of regular 1-forms. Throughout, we will assume that $b \in X(\mathbb{Q})$ and use it to embed X into J:

Definition 1.15. Let $A = \left\{\omega \in H^0(X, \Omega^1) : \text{ for all } P \in J(\mathbb{Q}),\ \langle P, \omega \rangle = 0\right\}$ be the subspace of *annihilating differentials*.

The embedding i induces an isomorphism of vector spaces $H^0(J_{\mathbb{Q}_p}, \Omega^1) \simeq H^0(X_{\mathbb{Q}_p}, \Omega^1)$ and we likewise have the pairing

$$\begin{aligned} J(\mathbb{Q}_p) \times H^0\left(J_{\mathbb{Q}_p}, \Omega^1\right) &\to \mathbb{Q}_p \\ (Q, \omega_J) &\mapsto \int_0^Q \omega_J, \end{aligned}$$

which induces a homomorphism

$$\log : J(\mathbb{Q}_p) \to H^0\left(J_{\mathbb{Q}_p}, \Omega^1\right)^*.$$

We thus have the following diagram:

(1)
$$\begin{array}{ccccc} X(\mathbb{Q}) & \longrightarrow & X(\mathbb{Q}_p) & & \\ \downarrow & & \downarrow & \searrow^{\mathrm{AJ_b}} & \\ J(\mathbb{Q}) & \longrightarrow & J(\mathbb{Q}_p) & \xrightarrow{\log} & H^0\left(J_{\mathbb{Q}_p}, \Omega^1\right)^* \simeq \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^* \end{array}$$

Remark 1.16. In general, since we are only considering the case of good reduction, we will identify the p-adic abelian integral on the Jacobian with the abelian integral given by p-adic integration on the curve. In the case of bad reduction (as discussed in Zureick-Brown's lecture course), there is a difference in the two integrals, as noted by Stoll [Sto19] and Katz–Rabinoff–Zureick-Brown [KRZB16]. See also the work of Besser–Zerbes [BZ] for a discussion of Vologodsky integration in the semistable case.

From now on, we will assume basic familiarity with rigid geometry, see for instance [Sch98, FvdP04].

Definition 1.17. Let X^{an} denote the rigid analytic space over $\mathbb{Q}_p$ associated to $X/\mathbb{Q}_p$. There is a specialization map from X^{an} to the reduction of X modulo p. The fibers of this map are called *residue disks*.

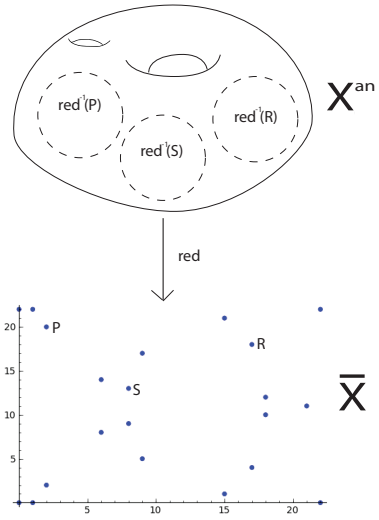


FIGURE 1. Residue disks in X^{an}

Corollary 1.18. *Let $X/\mathbb{Q}$ be a nice curve of genus g whose Jacobian has Mordell–Weil rank r less than g. Then $\#X(\mathbb{Q})$ is finite.*

Proof. If $X(\mathbb{Q}) = \varnothing$, then the statement is trivially true. Otherwise, fix a prime p of good reduction for X and fix $b \in X(\mathbb{Q})$ to define $i : X \to J$. Let A be the subspace of annihilating differentials. By additivity of integration pairing in the first argument, this condition is equivalent to requiring $\langle P_j, \omega \rangle = 0$ for a basis $\{P_j\}_{j=1}^r$ of the free part of $J(\mathbb{Q})$. So it leads to at most r linear constraints and $\dim(A) \geq g - r > 0$. Thus there is some $0 \neq \omega \in A$. Since $i(P) \in J(\mathbb{Q})$ for all $P \in X(\mathbb{Q})$ it follows that $\int_b^P \omega = 0$ for all $P \in X(\mathbb{Q})$. By Theorem 1.9 (7), the number of such P is finite in each residue disk of $X(\mathbb{Q}_p)$. Since the number of residue disks (i.e., $\#X(\mathbb{F}_p)$) is finite, the total number of points in $X(\mathbb{Q})$ is finite as well.

□

Remark 1.19. By *computing rational points via the Chabauty–Coleman method*, we mean that we compute the finite set of p-adic points

$$X(\mathbb{Q}_p)_1 := \left\{ z \in X(\mathbb{Q}_p) : \int_b^z \omega = 0 \text{ for } \omega \in A \right\}.$$

By construction, this set contains $X(\mathbb{Q})$. One potential difficulty is that $X(\mathbb{Q}_p)_1$ might be strictly larger than the set of known rational points, so more work must be done to provably extract $X(\mathbb{Q})$; see §2.3.2 for one approach to address this, known as the *Mordell–Weil sieve*.

We can use results about the number of zeros of p-adic power series (studied via Newton polygons) to refine the bound in the proof above. Combining this with Riemann–Roch gives Coleman's result, that for X satisfying the hypotheses of Corollary 1.18 and $p > 2g$ a good prime, we have (Theorem 1.4):

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$

Remark 1.20. Here are some related results:

(1) Lorenzini–Tucker [LT02] extended Coleman's result to the case where p is a prime of bad reduction.
(2) Stoll [Sto06] showed that one can choose the "best" ω for each residue disk, which improves the bound if $r < g$ and $p > 2r + 2$ is a good prime:
$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2r.$$
Stoll also showed that one can weaken the assumption that $p > 2r + 2$; if $p > 2$, then
$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2r + \left\lfloor \frac{2r}{p-2} \right\rfloor.$$
(3) Katz–Zureick-Brown [KZB13] extended Stoll's result to the case of bad reduction. If $p > 2g$ and $\mathcal{X}$ is the minimal proper regular model for X over $\mathbb{Z}_p$, then
$$\#X(\mathbb{Q}) \leq \#\mathcal{X}_{\mathrm{sm}}(\mathbb{F}_p) + 2r$$
where $\mathcal{X}_{\mathrm{sm}}(\mathbb{F}_p)$ is the set of smooth $\mathbb{F}_p$-rational points in the special fiber of $\mathcal{X}$.

As will be discussed in Zureick-Brown's lecture course, the Chabauty–Coleman method can be used to prove uniform bounds on the number of rational points on a nice curve. The first result along these lines was given by Stoll, for hyperelliptic curves:

Theorem 1.21 (Stoll [Sto19])**.** *Let $X/\mathbb{Q}$ be a hyperelliptic curve of genus g with Jacobian of Mordell–Weil rank r. If $r \leq g - 3$, then*

$$\#X(\mathbb{Q}) \leq 8rg + 33(g-1) - 1 \quad \text{if } r \geq 1 \quad \text{and} \quad \#X(\mathbb{Q}) \leq 33(g-1) + 1 \quad \text{if } r = 0\,.$$

This was generalized by Katz–Rabinoff–Zureick-Brown to nice curves:

Theorem 1.22 (Katz–Rabinoff–Zureick-Brown [KRZB16])**.** *If $X/\mathbb{Q}$ is a nice curve of genus g with $r \leq g-3$, then*

$$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28.$$

1.2. **The Chabauty–Coleman method and explicit Coleman integration.** Here we discuss how to construct an annihilating differential in the Chabauty–Coleman method, using explicit Coleman integration.

Example 1.23. Consider

$$X : y^2 = x^5 - 2x^3 + x + \frac{1}{4},$$

which has LMFDB label 971.a.971.1 [LMF20b]. Here are some facts about this curve:

- Searching for rational points in a box, we find that the set of rational points $X(\mathbb{Q})$ contains $\{\infty, (0, \pm 1/2), (-1, \pm 1/2), (1, \pm 1/2)\}$.
- The Jacobian is simple, and its Mordell–Weil group has the structure $J(\mathbb{Q}) \cong \mathbb{Z}$. The point
$$[(-1,-1/2) - (0,1/2)] \in J(\mathbb{Q})$$
has infinite order, as can be seen by computing Coleman integrals on regular 1-forms (see below).
- The conductor N is 971, which is prime. So X has good reduction at $p = 3$, and we compute that $\#X(\mathbb{F}_3) = 7$. Using Stoll's refinement of the Chabauty–Coleman bound gives
$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_3) + 2 \cdot 1 + \left\lfloor \frac{2 \cdot 1}{3-2} \right\rfloor = 11,$$
so this bound by itself will not prove[4] that we have all of the $\mathbb{Q}$-points.

Our strategy will be to use $p = 3$ to construct an annihilating differential. Since X is hyperelliptic of genus 2, a basis of $\mathrm{H}^0(X_{\mathbb{Q}_3}, \Omega^1)$ is

$$\left\{ \omega_i = \frac{x^i dx}{2y} \right\}_{i=0,1}.$$

So the annihilating differential η is a $\mathbb{Q}_3$-linear combination of ω_0 and ω_1. We will use the values of

$$\int_{(0,1/2)}^{(-1,-1/2)} \omega_i$$

to compute η.

We can do this in `SageMath` as follows:

```
R.<x> = QQ[]
X = HyperellipticCurve(x^5-2*x^3+x+1/4)
p = 3
K = Qp(p,15)
XK = X.change_ring(K)
XK.coleman_integrals_on_basis(XK(0,1/2),XK(-1,-1/2)) #basis is {x^i*dx/(2y)}, i = 0,...,3
(3 + 3^2 + 3^4 + 3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 3^10 + O(3^11),
2 + 2*3 + 2*3^3 + 3^4 + 3^6 + 2*3^8 + 2*3^9 + O(3^10),
2*3^-1 + 2*3 + 2*3^2 + 3^3 + 3^5 + 3^6 + 3^7 + O(3^9),
2*3^-2 + 3^-1 + 2 + 2*3 + 3^2 + 2*3^3 + 3^4 + 2*3^5 + 2*3^6 + 2*3^7 + O(3^8))
```

[4]Note that we do not yet know that we have all of the $\mathbb{Q}$-points, but we suspect we do. We would like to prove this.

We find that

$$\alpha := \int_{(0,1/2)}^{(-1,-1/2)} \omega_0 = 3 + 3^2 + 3^4 + 3^5 + 2\cdot 3^6 + 2\cdot 3^7 + 2\cdot 3^8 + 3^{10} + O(3^{11}),$$
$$\beta := \int_{(0,1/2)}^{(-1,-1/2)} \omega_1 = 2 + 2\cdot 3 + 2\cdot 3^3 + 3^4 + 3^6 + 2\cdot 3^8 + 2\cdot 3^9 + O(3^{10}).$$

With a slightly different choice of basis[5], we can also do these computations in `Magma` (using the package [BTb], available on GitHub) as follows:

```
> load "coleman.m";
> data:=coleman_data(y^2-(x^5-2*x^3+x+1/4),3,10);
> P1:= set_point(0,1/2, data);
> P2:= set_point(-1,-1/2,data);
> coleman_integrals_on_basis(P1,P2,data); //8 times the integrals above
(-7609*3 + O(3^10) 13537 + O(3^10) 77056*3^-1 + O(3^10) -6512*3^-2 + O(3^10))
```

So $\int_{(0,1/2)}^{(-1,-1/2)} \beta\omega_0 - \alpha\omega_1 = 0$, and we take

$$\eta = \beta\omega_0 - \alpha\omega_1$$

as our annihilating differential.

In order to use η to compute $X(\mathbb{Q})$, or more precisely the finite set $X(\mathbb{Q}_3)_1$, that by construction, contains $X(\mathbb{Q})$, we next compute the collection of "indefinite" Coleman integrals

$$\left\{ \int_{(0,1/2)}^{P_t} \eta \right\}$$

where P_t ranges over all residue disks, and solve for all $z \in X(\mathbb{Q}_3)$ such that $\int_{(0,1/2)}^{z} \eta = 0$. Note that to compute these indefinite Coleman integrals, we can take P_0 a lift of an $\mathbb{F}_3$-point in the same residue disk as P_t. Then

$$\int_{(0,1/2)}^{P_t} \eta = \int_{(0,1/2)}^{P_0} \eta + \int_{P_0}^{P_t} \eta$$

where the first is some 3-adic constant, and the latter is a tiny integral computed using a power series. So to compute α, β and $\int_{(0,1/2)}^{P_0} \eta$, we need to compute Coleman integrals between points not in the same residue disk.

Now we explain how to compute these integrals on the curve, using the action of Frobenius on p-adic cohomology.

Remark 1.24. Before we go on, we should note that there is a standard alternative approach to the one presented below for computing Coleman integrals of regular 1-forms between points not in the same residue disk that goes as follows.

Suppose we want to compute the Coleman integral $\int_P^Q \omega$, where $P, Q \in X(\mathbb{Q}_p)$. Letting J denote the Jacobian of X, we first compute a non-zero integer k such that the point $k(P-Q)$ is trivial in $J(\mathbb{F}_p)$: for instance, we could take k to be the order of $J(\mathbb{F}_p)$. Then computing $D := [k(P-Q)]$ as an element in the residue disk at 0 of $J(\mathbb{Q}_p)$, we can rewrite the integral as a sum of tiny integrals over D, and then use $\int_{[P-Q]} \omega = \frac{1}{k}\int_D \omega$.

[5]In this example, the `Magma` basis is the `SageMath` basis rescaled by a factor of 8.

This has worked well in a number of examples in the literature, though there are a few potential limitations. First, implementations of Jacobian arithmetic over $\mathbb{Q}_p$ are currently restricted to very special curves, such as those that are hyperelliptic. Secondly, while the Chabauty–Coleman method only uses integrals of regular 1-forms, there are other applications for which integrals of forms of the second or third kind are useful. Moreover, since this approach uses properties of the Jacobian, it does not have an obvious generalization to iterated integrals. So from the perspective of the nonabelian Chabauty method, where iterated integration is needed, we present the following approach.

We will integrate over a wide open subspace of X^{an}:

Definition 1.25. A *wide open subspace* of X^{an} is the complement in X^{an} of the union of a finite collection of disjoint closed disks of radius $\lambda_i < 1$.

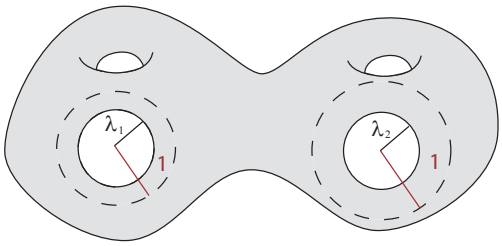


FIGURE 2. A wide open subspace of X^{an}

Here are some further properties of Coleman integrals that we will need:

Theorem 1.26 (Coleman [Col85b])**.** *Let η and ξ be 1-forms on a wide open subspace V of X^{an}, and $P, Q, R \in V(\overline{\mathbb{Q}}_p)$ that are not poles of η or ξ. Let $a, b \in \overline{\mathbb{Q}}_p$. The definite Coleman integral has the following properties:*

(1) Linearity:

$$\int_P^Q (a\eta + b\xi) = a\int_P^Q \eta + b\int_P^Q \xi.$$

(2) Additivity in endpoints:

$$\int_P^Q \eta = \int_P^R \eta + \int_R^Q \eta.$$

(3) Change of variables: if $V' \subset X'$ is a wide open subspace of a rigid analytic space X', ω' a 1-form on V' and $\phi : V \to V'$ a rigid analytic map, then

$$\int_P^Q \phi^*\omega' = \int_{\phi(P)}^{\phi(Q)} \omega'.$$

(4) Fundamental Theorem of Calculus:

$$\int_P^Q df = f(Q) - f(P)$$

for f a rigid analytic function on V.

(5) Galois compatibility: If $P, Q \in V(\mathbb{Q}_p)$ and ω is defined over $\mathbb{Q}_p$, then $\int_P^Q \omega \in \mathbb{Q}_p$.

We would first like to integrate $\int_P^Q \omega$ for ω a 1-form of the second kind, where $P, Q \in V(\mathbb{Q}_p)$. We first discuss how to do this in the case when X is a *hyperelliptic* curve and then present a more general construction in §1.4. (In our discussion of p-adic heights in §2.2, we will also describe how to compute integrals of forms of the third kind.)

The idea is to do the following:

(1) Take ϕ to be a lift of p-power Frobenius from the special fiber.
(2) Compute a basis $\{\omega_i\}$ of 1-forms of the second kind.
(3) Compute $\phi^*\omega_i$ via Kedlaya's zeta function algorithm [Ked01, Ked03] and use properties of Coleman integrals to relate $\int_P^Q \phi^*\omega_i$ to $\int_P^Q \omega_i$ and other terms we can compute.
(4) Use linear algebra to solve for $\int_P^Q \omega_i$.

To do this, we introduce some p-adic cohomology as in Kedlaya's algorithm. For further details, two standard references for rigid analytic geometry are the books by Fresnel and van der Put [FvdP04] and Bosch, Güntzer, and Remmert [BGR84]. See also [Edi] for a nice exposition by Edixhoven of Kedlaya's algorithm.

1.3. **Some p-adic cohomology.** In [Ked01], Kedlaya gave an algorithm to compute the zeta function of a hyperelliptic curve over a finite field, using Monsky–Washnitzer cohomology. Here is a brief outline of Kedlaya's algorithm:

(1) Work in an affine piece of the hyperelliptic curve, given by deleting Weierstrass points.
(2) Take ϕ to be a lift of p-power Frobenius from the special fiber, sending $x \mapsto x^p$ and Hensel lifting to find the image of y.
(3) Compute the action of Frobenius on a basis of de Rham cohomology (of a lift of the curve) and reduce the pole order of each resulting differential using relations in cohomology.

It turns out that Kedlaya's algorithm produces a few other outputs that can be assembled into an algorithm for Coleman integration on hyperelliptic curves, as given by Balakrishnan–Bradshaw–Kedlaya [BBK10]. In this section, we give an overview of Kedlaya's algorithm and the corresponding Coleman integration algorithm.

For simplicity, we will assume that we start with a genus g hyperelliptic curve $\tilde{X}$ defined over $\mathbb{Q}$, given by $y^2 = \tilde{P}(x)$, where $\tilde{P}(x)$ is a monic polynomial of degree $2g+1$. Let $p \neq 2$ be a prime at which $\tilde{X}$ has good reduction, and consider $\overline{X}/\mathbb{F}_p$, with affine equation $y^2 = P(x)$. Take $X = \overline{X} \backslash \{\infty, y = 0\}$. (We explain why we work with this particular affine curve below.)

Let $A = \mathbb{Z}_p[x, y, y^{-1}]/(y^2 - \tilde{P}(x))$. First, we form the weak completion $A^\dagger$ of A, which can be described as follows. Let v_p denote the p-adic valuation on $\mathbb{Z}_p$, and extend it to polynomials by $v_p(\sum a_i x^i) = \min_i\{v_p(a_i)\}$. The elements of $A^\dagger$ can then be described as the series

$$\sum_{n=-\infty}^{\infty} (S_n(x) + T_n(x)y)y^{2n}$$

where the S_n and T_n are polynomials of degree at most $2g$ such that

$$\liminf_{n\to\infty} \frac{v_p(S_n)}{n}, \qquad \liminf_{n\to\infty} \frac{v_p(S_{-n})}{n}, \qquad \liminf_{n\to\infty} \frac{v_p(T_n)}{n}, \qquad \liminf_{n\to\infty} \frac{v_p(T_{-n})}{n}$$

are all positive.

Monsky–Washnitzer cohomology is a p-adic cohomology theory which takes smooth affine varieties over fields of characteristic $p > 0$ as input, and outputs finite-dimensional $\mathbb{Q}_p$-vector spaces. There is a comparison theorem due to the work of Berthelot [Ber97, Prop. 1.10] (comparing Monsky–Washnitzer

and rigid cohomology) and Baldassarri–Chiarellotto [BC94, Cor. 2.6] (comparing rigid cohomology with de Rham cohomology), which relates Monsky–Washnitzer cohomology groups with algebraic de Rham cohomology groups:

Theorem 1.27 (Special case of Baldassarri–Chiarellotto and Berthelot)**.** *Let Y be a smooth affine variety over $\mathbb{F}_p$ and $\widetilde{Y}$ a smooth affine variety over $\mathbb{Q}_p$ that is a lift of Y. (Since Y is smooth affine, such lifts always exist.) Then the Monsky–Washnitzer cohomology of Y coincides with the algebraic de Rham cohomology of $\widetilde{Y}$:*

$$\mathrm{H}^1_{\mathrm{dR}}(\widetilde{Y}) = \mathrm{H}^1_{\mathrm{MW}}(Y).$$

The Monsky–Washnitzer cohomology groups are equipped with an action of Frobenius, hence Theorem 1.27 tells us that we can compute the action of Frobenius on de Rham cohomology.

Proposition 1.28. *The first de Rham cohomology of A splits into two eigenspaces under the hyperelliptic involution*

$$X \to X,\ (x,y) \mapsto (x,-y).$$

The first eigenspace $\mathrm{H}^1(A)^+$ is the positive eigenspace generated by

$$\left\{ \frac{x^i dx}{y^2} : i = 0, \dots, 2g \right\},$$

and the second eigenspace $\mathrm{H}^1(A)^-$ is the negative eigenspace generated by

$$\left\{ \frac{x^i dx}{y} : i = 0, \dots, 2g-1 \right\},$$

Moreover, passing to $A^\dagger$ does not change the cohomology, and we compute the action of Frobenius on $\mathrm{H}^1(A^\dagger)^-$. We lift p-power Frobenius to an endomorphism σ of $A^\dagger$ in the following manner: On polynomials in $\mathbb{Z}_p[x]$, we send

$$\sigma : x \mapsto x^p. \tag{2}$$

Since $y^2 = \widetilde{P}(x)$ inside A and $A^\dagger$, we see that the action of σ on y must satisfy the following:

$$(y^\sigma)^2 = (y^2)^\sigma = (\widetilde{P}(x))^\sigma = \widetilde{P}(x)^\sigma \left(\frac{y^2}{\widetilde{P}(x)} \right)^p = \frac{y^{2p} \widetilde{P}(x)^\sigma}{\widetilde{P}(x)^p}.$$

We have

$$\sigma : y \mapsto y^p \left(\frac{\widetilde{P}(x)^\sigma}{\widetilde{P}(x)^p} \right)^{\frac{1}{2}} = y^p \left(1 + \frac{\widetilde{P}(x)^\sigma - \widetilde{P}(x)^p}{\widetilde{P}(x)^p} \right)^{\frac{1}{2}},$$

and by using a Taylor expansion for $(1+\cdot)^{-\frac{1}{2}}$, we get an identity

$$\frac{1}{y^\sigma} = \frac{1}{y^p} \sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{j} \left(\frac{\widetilde{P}(x)^\sigma - \widetilde{P}(x)^p}{\widetilde{P}(x)^p} \right)^j = \frac{1}{y^p} \sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{j} \left(\frac{\widetilde{P}(x)^\sigma - \widetilde{P}(x)^p}{y^{2p}} \right)^j. \tag{3}$$

The reason we write the expansion for $\frac{1}{y^\sigma}$ in this way is to see the p-adic convergence, since $\widetilde{P}(x)^\sigma - \widetilde{P}(x)^p$ is divisible by p, so as $j \to \infty$, the summands go to 0.

This expansion will be used below, and perhaps now it is more clear why we removed Weierstrass points from our curve: given our choice of Frobenius lift, we cannot divide by y.

Finally, we extend the p-power Frobenius action to differentials by sending

$$\sigma^* : dx \mapsto d(x^p) = p x^{p-1} dx. \tag{4}$$

In order to prove Proposition 1.28, we will need two key reduction lemmas to compute $(\frac{x^i dx}{y})^\sigma$.

Lemma 1.29 (Kedlaya [Ked01, p. 5]). *If* $R(x) = \widetilde{P}(x)B(X) + \widetilde{P}'(x)C(X)$, *then*

$$\frac{R(x)dx}{y^s} = \left(B(x) + \frac{2C'(x)}{s-2}\right)\frac{dx}{y^{s-2}} \tag{5}$$

as elements in $\mathrm{H}^1_{\mathrm{MW}}(X)$.

Also, using $y^2 = \widetilde{P}(x)$, we have $d(y^2) = d\widetilde{P}(x)$, so $2ydy = \widetilde{P}'(x)dx$. This gives us

$$dy = \frac{\widetilde{P}'(x)dx}{2y}. \tag{6}$$

This allows us to compute:

$$\begin{aligned} d(x^iy^j) =& ix^{i-1}y^j dx + x^i j y^{j-1} dy \\ \overset{(6)}{=}& ix^{i-1}y^j dx + jx^i y^{j-1}\frac{\widetilde{P}'(x)dx}{2y} = (2ix^{i-1}y^{j+1} + jx^i\widetilde{P}'(x)y^{j-1})\frac{dx}{2y} \end{aligned}$$

(So the *highest* monomial of $d(x^iy^j)$ is $x^{i-1}y^{j+1}$ if $1 \le i < 2g+1$ and $x^{2g}y^{j-1}$ if $i = 0$. The *lowest* monomial of $d(x^iy^j)$ is of the form x^ky^{j-1} with $0 \le k < 2g+1$.) As a special case of this computation, we have

$$\begin{aligned} d(2Q(x)y) &= 2Q(x)dy + 2Q'(x)ydx \\ &\overset{(6)}{=} 2Q(x)\frac{\widetilde{P}'(x)dx}{2y} + 2Q'(x)ydx \\ &\overset{y^2=\widetilde{P}(x)}{=} (Q(x)\widetilde{P}'(x) + 2Q'(x)\widetilde{P}(x))\frac{dx}{y}, \end{aligned}$$

proving the second reduction lemma:

Lemma 1.30 (Kedlaya [Ked01, p. 5]). *If* $Q(x) = x^{m-2g}$, *then*

$$d(2Q(x)y) = (Q(x)\widetilde{P}'(x) + 2Q'(x)\widetilde{P}(x))\frac{dx}{y} = 0 \tag{7}$$

as elements in $\mathrm{H}^1_{\mathrm{MW}}(X)$.

To compute $(\frac{x^i dx}{y})^\sigma$, we expand using (2), (3), (4) and reduce using the relations (5) and (7). The reduction process is subtracting appropriate linear combinations of $d(x^iy^j)$ and using the relationship $y^2 = \widetilde{P}(x)$.

The relation

$$\left(\frac{x^i dx}{y}\right)^\sigma = \frac{1}{y^\sigma}x^{pi}px^{p-1}dx$$

plus (3) gives an infinite sum

$$\left(\frac{x^i dx}{y}\right)^\sigma = \frac{px^{pi+p-1}}{y^p}\sum_{j=0}^{\infty}\binom{-\frac{1}{2}}{j}\left(\frac{\widetilde{P}(x)^\sigma - \widetilde{P}(x)^p}{y^{2p}}\right)^j dx. \tag{8}$$

To implement the expansion and reduction on a computer, we have to take a truncation of this infinite sum, and thus we need to know how many terms we need to take to get a provably correct result (more on this in a minute). Suppose we have computed this precision and the result in (8) is

$$\sum_{j=-L_1}^{L_2}\frac{R_j(x)dx}{y^{2j+1}}. \tag{9}$$

Here is how we use the reduction relations: we eliminate the $j = L_2$ term, then $j = L_2 - 1$ term. Iterate this procedure until no terms with $j > 0$ remain. Repeat the same thing for $j = -L_1, -(L_1 - 1), \ldots$ terms. At the end of this reduction algorithm, we will be left with

$$\left(\frac{x^i dx}{y}\right)^\sigma = dh_i + \sum_{j=0}^{2g-1} M_{ji} \frac{x^j dx}{y},$$

and as $dh_i \sim 0$ in cohomology, this gives us the matrix of Frobenius M.

Precision is lost when we divide by p in the reduction algorithm. We need to measure the loss of precision at each step to know how many provably correct digits we have. Let $R(x) \in \mathbb{Z}_p[x]$ be a polynomial of degree at most $2g$ and $m \geq 0$.

By (5), the reduction of

$$\omega := R(x)\frac{dx}{y^{2m+1}}$$

is $\omega = B(x)\frac{dx}{y} + df$ for some $B(x) \in \mathbb{Q}_p[x]$ with degree at most $2g-1$ and $f = \sum_{j=-1}^{m-1} \frac{F_k(x)}{y^{2k+1}}$ with each F_k having degree at most $2g$. The first precision result is:

Lemma 1.31 ([Ked01, Lemma 2], [Edi, §4.3.4]). *In the above setting*[6]*, we have*

$$p^{\lfloor \mathrm{Log}_p(2m-1)\rfloor} B(x) \in \mathbb{Z}_p[x].$$

By (7), the reduction of

$$\omega := \frac{R(x)y^{2m}dx}{y}$$

is $\omega = B(x)\frac{dx}{y} + df$ for some $B(x) \in \mathbb{Q}_p[x]$ with degree at most $2g-1$, and

$$f = Cy^{2m+1} + \sum_{k=0}^{m-1} F_k(x) y^{2k+1}$$

with $C \in \mathbb{Q}_p$ and each F_k having degree at most $2g$.

Lemma 1.32 ([Ked03]). *In the above setting, we have*

$$p^{\lfloor \mathrm{Log}_p((2g+1)(2m+1))\rfloor} B(x) \in \mathbb{Z}_p[x].$$

Putting Lemmas 1.31 and 1.32 together, one gets the following:

Proposition 1.33 ([Cha16, p. 34]). *To get N correct digits in the matrix of Frobenius M, we start with precision*

$$N_1 = N + \max\{\lfloor \mathrm{Log}_p(2N_2 - 3)\rfloor, \lfloor \mathrm{Log}_p(2g+1)\rfloor\} + 1 + \lfloor \mathrm{Log}_p(2g-1)\rfloor,$$

in which N_2 is the smallest integer such that

$$N_2 - \max\{\lfloor \mathrm{Log}_p(2N_2 + 1), \lfloor \mathrm{Log}_p(2g+1)\rfloor\} \geq N.$$

In particular, in (8), we take the truncation

$$\left(\frac{x^i dx}{y}\right)^\sigma = \frac{px^{pi+p-1}}{y^p} \sum_{j=0}^{N_2-1} \binom{-\frac{1}{2}}{j} \left(\frac{\widetilde{P}(x)^\sigma - \widetilde{P}(x)^p}{y^{2p}}\right)^j dx.$$

[6] In this section, Log_p will denote the base p logarithm, to disambiguate from $\log_p$ in subsequent sections, which will denote the p-adic logarithm.

Algorithm 1.34 (Kedlaya's algorithm)**.**
Input:

- The basis of differentials $\{\omega_i = x^i dx/y\}_{i=0}^{2g-1}$ of $\mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})$ for a genus g hyperelliptic curve X given by a monic odd degree model, with good reduction at p.
- The desired precision N.

Output: The $2g \times 2g$ matrix M of a p-power lift of Frobenius ϕ, as well as functions $h_i \in A^\dagger$ such that $\phi^*(\omega_i) = dh_i + \sum_{j=0}^{2g-1} M_{ij}^{\intercal}\omega_j$ to precision $O(p^N)$

(1) Compute the working precision N_1 as in Proposition 1.33, so that all computations will be done mod p^{N_1}.
(2) For each i, compute $F_i := \phi^*(\omega_i)$ and group the resulting terms as $(\sum p^{k+1} c_{i,k,j} y^j)dx/y$, where the $c_{i,k,j} \in \mathbb{Z}_p[x]$ have degree less than or equal to $2g+1$.
(3) Compute a list of differentials $d(x^i y^j)$, where $0 \leq i < 2g+1$ and $j \equiv 1 \pmod 2$.
(4) If F_i has a term $(x^i y^j)dx/y$ with $j < 0$, consider the term $(c_{i,k,j} y^j)dx/y$ where j is minimal. Take the unique linear combination of the $d(x^k y^{1+j})$ such that when this linear combination is subtracted off of F_i and re-initialize this as F_i. Do this until F_i no longer has terms of the form $(x^m y^j)dx/y$ with $j < 0$.
(5) If F_i has terms with $j \geq 0$, let $(x^m y^j)dx/y$ be the term with the highest monomial of F_i. Let $(x^k y^l)dx/y$ be the term such that $d(x^k y^l)$ has highest term $(x^m y^j)dx/y$ and subtract off the appropriate multiple of $d(x^k y^l)$ such that the resulting sum no longer has terms of the form $(x^m y^j)dx/y$ with $j \geq 0$. Re-initialize this as F_i and repeat this process until the resulting F_i is of the form $\left(M_{0i} + M_{1i}x \ + \cdots + M_{2g-1i}x^{2g-1}\right) dx/y$.
(6) For each i, return the expression

$$\phi^*(\omega_i) = dh_i + \sum_{j=0}^{2g-1} M_{ij}^{\intercal}\omega_j.$$

Remark 1.35. Analyzing p-adic precision is a delicate task. We illustrate this in one example found by Chan [Cha16] below, where the previously published bounds contained a small inaccuracy. For the remainder of these notes, we do not say much more about p-adic precision analysis of the relevant constructions and instead give relevant pointers to the literature. We encourage the reader to keep the issue of p-adic precision in mind as they work through the algorithms.

Example 1.36 ([Cha16, Remark 13]). Consider the elliptic curve over $\mathbb{Q}$ defined by

$$y^2 = \widetilde{P}(x) = x^3 + x + 1.$$

This curve has good reduction at the prime $p = 5$. We wish to obtain $N = 2$ correct digits of expansion. Proposition 1.33 tells us that taking $N_2 = N_1 = 3$ suffices. Consider the two differentials $\frac{dx}{y}, \frac{xdx}{y}$. We expand (8) and use the equation $y^2 = \widetilde{P}(x)$ as needed to reduce the degree in x in the numerators to

produce the following:

$$\left(\frac{dx}{y}\right)^\sigma = \left(\frac{25x+50}{y^{15}} + \frac{75x^2+100x+25}{y^{13}} + \frac{50x^2+50x+100}{y^{11}} + \frac{75x+50}{y^9} + \frac{50x^2+50x}{y^7} + \frac{70x^2+70x+25}{y^5} + \frac{5x}{y^3}\right)dx \pmod{5^3},$$

$$\left(\frac{xdx}{y}\right)^\sigma = \left(\frac{100x^2+100x+75}{y^{15}} + \frac{25x^2+50x+75}{y^{13}} + \frac{50x^2+100x+100}{y^{11}} + \frac{25x^2+75x+75}{y^9} + \frac{75x^2+100}{y^7} + \frac{85x^2+90+50}{y^5} + \frac{15x^2+30x+85}{y^3} + \frac{5x^3+65x+65}{y}\right)dx \pmod{5^3}.$$

Let F_k denote the polynomial in x in the numerator in each of the summands: i.e., writing them as $\frac{F_k(x)dx}{y^{2k+1}}$ modulo 5^3. Compute the sequence S_k for $k = 7, 6, \cdots, 0$ inductively by first setting $S_7 = F_7$, and afterwards, given S_{k+1}, find polynomials B_{k+1}, C_{k+1} such that $S_{k+1} = B_{k+1}\widetilde{P} + C_{k+1}\widetilde{P}'$, and then set $S_k(x) = F_k(x) + B_{k+1}(x) + \frac{2C'_{k+1}(x)}{2k+1}$. Carrying this out, one finds

$$\left(\frac{dx}{y}\right)^\sigma = 15x\frac{dx}{y} \pmod{5^2}$$

$$\left(\frac{xdx}{y}\right)^\sigma = (22x+18)\frac{dx}{y} \pmod{5^2}$$

This gives us the matrix of the 5-power Frobenius

$$\begin{pmatrix} 0 & 18 \\ 15 & 22 \end{pmatrix} \pmod{5^2},$$

with $N = 2$ correct digits of expansion. Note that taking $N_1 = 3$ is necessary as well, as taking $N_1 = 2$ instead gives the matrix

$$\begin{pmatrix} 15 & 18 \\ 0 & 22 \end{pmatrix} \pmod{5^2}.$$

Now here is the application to Coleman integration, as carried out by Balakrishnan–Bradshaw–Kedlaya [BBK10]. Below we let ϕ denote the lift of p-power Frobenius described earlier.

Algorithm 1.37 (Coleman integration on a hyperelliptic curve [BBK10])**.**
Input:

- A prime $p > 2$ of good reduction for a hyperelliptic curve X
- Points $P, Q \in X(\mathbb{Q}_p)$ not contained in a Weierstrass residue disk
- A 1-form ω of the second kind

Output: The Coleman integral $\int_P^Q \omega$.

(1) Since ω is of the second kind, we may write it as a linear combination of a basis $\{\omega_i\}_{i=0}^{2g-1}$ for $\mathrm{H}^1_{\mathrm{dR}}(X)$ together with an exact form. Use Kedlaya's algorithm to write $\omega = dh + \sum_{i=0}^{2g-1} a_i\omega_i$, which allows us to specialize to the case of Coleman integrals of basis differentials.
(2) Use Kedlaya's algorithm to write, for each basis differential ω_i, the reduced form

$$\phi^*\omega_i = dh_i + \sum_{j=0}^{2g-1} M_{ji}\omega_j.$$

(3) Using properties of the Coleman integral, we have

$$\begin{pmatrix} \vdots \\ \int_P^Q \omega_j \\ \vdots \end{pmatrix} = (M^{\intercal} - I)^{-1} \begin{pmatrix} \vdots \\ h_i(P) - h_i(Q) - \int_P^{\phi(P)} \omega_i - \int_{\phi(Q)}^Q \omega_i \\ \vdots \end{pmatrix}. \tag{10}$$

(4) Compute $\int_P^Q \omega = h(Q) - h(P) + \sum_{i=0}^{2g-1} a_i \int_P^Q \omega_i$.

Remark 1.38. We derive (3) above using the following:

$$\begin{aligned} \int_{\phi(P)}^{\phi(Q)} \omega_i &= \int_P^Q \phi^* \omega_i \\ \text{(by Kedlaya)} &= \int_P^Q dh_i + \sum_{j=0}^{2g-1} M_{ji}\omega_j \\ &= \int_P^Q dh_i + \sum_{j=0}^{2g-1} M_{ji} \int_P^Q \omega_j \\ &= h_i(Q) - h_i(P) + \sum_{j=0}^{2g-1} M_{ji} \int_P^Q \omega_j. \end{aligned}$$

By the additivity of the Coleman integral on endpoints, we get

$$\int_P^{\phi(P)} \omega_i + \int_{\phi(P)}^{\phi(Q)} \omega_i + \int_{\phi(Q)}^Q \omega_i = \int_P^{\phi(P)} \omega_i + \int_{\phi(Q)}^Q \omega_i + h_i(Q) - h_i(P) + \sum_{j=0}^{2g-1} M_{ji} \int_P^Q \omega_j.$$

The left hand side of the equality becomes $\int_P^Q \omega_i$. For the right hand side, P and $\phi(P)$ are in the same residue disk, making $\int_P^{\phi(P)} \omega_i$ a tiny integral and therefore computable via its power series expansion. The same is true for the pair $\phi(Q)$ and Q. The h_i are given to us explicitly from Kedlaya's algorithm, and we can evaluate them on Q and P. Notice that $M^{\intercal} - 1$ is invertible since, by the Weil conjectures, the eigenvalues of M have norm $\sqrt{p} \neq 1$. Therefore we can compute the left hand side by solving the linear equation.

Remark 1.39. For Weierstrass residue disks, the lift of Frobenius is not defined over the entirety of the disc, but due to overconvergence it is defined near the boundary of the residue disk. So if W is a Weierstrass point and we would like to compute $\int_W^P \omega_i$, we choose a point S close to the boundary of the Weierstrass disk of W and decompose the integral as

$$\int_W^P \omega_i = \int_W^S \omega_i + \int_S^P \omega_i.$$

On the right hand side, the first term is a tiny integral while the second term can be computed using the above method. However, this is computationally expensive, as we have to work over a totally ramified extension of $\mathbb{Q}_p$ to compute the integral.

Remark 1.40. For precision estimates in Algorithm 1.37, see [BBK10, §4.1]. Roughly speaking, there is some loss of precision from truncations of power series giving the necessary tiny integrals, as well as from the valuation of the determinant of the matrix $M^{\intercal} - 1$.

Remark 1.41. In the case of p-adic integration for a *bad* prime p, Katz and Kaya [KK22] recently gave an algorithm to compute p-adic abelian integrals on hyperelliptic curves. They do this by covering a

hyperelliptic curve with bad reduction at p by annuli and basic wide open sets, and then reduce the computation of Berkovich–Coleman integrals to the known algorithms for integration of a 1-form of the second or third kind on a hyperelliptic curve with good reduction [BBK10, BB12] and to integration in annuli.

Remark 1.42. For a genus g hyperelliptic curve over $\mathbb{F}_{p^n}$, Kedlaya's algorithm computes the matrix of p-power Frobenius mod p^N in time $\widetilde{O}(pN^2g^2n)$, where $\widetilde{O}(X)$ denotes $O(X(\log X)^k)$ for some $k \geq 0$. Harvey [Har07] showed that one could interpret the reductions in cohomology in terms of linear recurrences to reduce the dependence on p in the runtime of the algorithm to $\sqrt{p}$. This was later generalized by Minzlaff [Min10] to superelliptic curves. Best showed that similar ideas can be used to improve the runtime of Coleman integration algorithms, first in the case of hyperelliptic curves over $\mathbb{Q}_p$ [Bes19] and superelliptic curves over unramified extensions of $\mathbb{Q}_p$ [Bes21].

Now we return to the Chabauty–Coleman method for a nice curve $X/\mathbb{Q}$.

Example 1.43. Recall the set-up of Example 1.23, with the genus 2 curve

$$X : y^2 = x^5 - 2x^3 + x + \frac{1}{4},$$

with known rational points

$$X(\mathbb{Q})_{\text{known}} = \{\infty, (0, \pm 1/2), (-1, \pm 1/2), (1, \pm 1/2)\}.$$

We computed an annihilating differential

$$\eta = \beta\omega_0 - \alpha\omega_1,$$

where

$$\alpha := \int_{(0,1/2)}^{(-1,-1/2)} \omega_0 = 3 + 3^2 + 3^4 + 3^5 + 2\cdot 3^6 + 2\cdot 3^7 + 2\cdot 3^8 + 3^{10} + O(3^{11}),$$
$$\beta := \int_{(0,1/2)}^{(-1,-1/2)} \omega_1 = 2 + 2\cdot 3 + 2\cdot 3^3 + 3^4 + 3^6 + 2\cdot 3^8 + 2\cdot 3^9 + O(3^{10}),$$

and these values of α and β were produced using Algorithm 1.37.

Now we would like to determine the set

$$X(\mathbb{Q}_3)_1 := \{z \in X(\mathbb{Q}_3) : \int_{(0,1/2)}^{z} \eta = 0\} \supset X(\mathbb{Q}).$$

We begin by enumerating the points in $X(\mathbb{F}_3)$:

$$X(\mathbb{F}_3) = \{\infty, (0, \pm 1), (1, \pm 1), (2, \pm 1)\},$$

which indexes the residue disks. Now we would like to compute the power series expansions of the collection of "indefinite" Coleman integrals $\left\{\int_{(0,1/2)}^{P_t} \eta\right\}$, where P_t ranges over all residue disks, and solve for all $z \in X(\mathbb{Q}_3)$ such that $\int_{(0,1/2)}^{z} \eta = 0$. Note that to compute these indefinite Coleman integrals, we can take P_0 a lift of an $\mathbb{F}_3$-point in the same residue disk as P_t. Then

$$\int_{(0,1/2)}^{P_t} \eta = \int_{(0,1/2)}^{P_0} \eta + \int_{P_0}^{P_t} \eta, \tag{11}$$

where the first integral on the right-hand side of (11) is some 3-adic constant, and the second is a tiny integral computed using a local coordinate at P_0. However, since each residue disk contains one rational point, we may take P_0 to be the rational point in the residue disk. This sets the constant of integration

to 0 in each disk, by construction of the annihilating differential. Thus the computation is now purely local. Moreover, using the hyperelliptic involution, we need only to consider the residue disk of P_0 and not the disk of $i(P_0)$ as well.

So we carry out the computation in the residue disks of ∞, (0,1/2), (1,1/2), and $(-1, 1/2)$. For instance, in the residue disk of $(1, 1/2)$, a local coordinate is given by

$$x(t) = 1 + t + O(t^{20})$$

$$y(t) = \frac{1}{2} + 4t^2 + 8t^3 - 11t^4 - 63t^5 + 24t^6 + 680t^7 + 695t^8 - 7210t^9 - 19881t^{10} + 64544t^{11} + 374802t^{12} - 301946t^{13} - 5872722t^{14} - 5265422t^{15} + 78467963t^{16} + 210631116t^{17} - 840861878t^{18} - 4667976084t^{19} + O(t^{20})$$

and letting $I(t) := \int_{(0,1/2)}^{P_t} \eta = \int_{P_0}^{P_t} \eta$, the power series for $I(3t)$ is given by

$$\left(2 + 3 + 2\cdot 3^2 + 3^3 + 2\cdot 3^5 + 3^6 + 2\cdot 3^8 + 3^9 + O(3^{10})\right) t + \left(3^2 + 2\cdot 3^3 + 2\cdot 3^4 + 2\cdot 3^6 + 3^7 + 3^8 + 3^9 + 2\cdot 3^{10} + O(3^{12})\right) t^2 +$$
$$\left(2\cdot 3 + 3^2 + 3^5 + 3^7 + 3^8 + 3^{10} + O(3^{11})\right) t^3 + \left(3^3 + 3^4 + 3^5 + 2\cdot 3^6 + 2\cdot 3^7 + 2\cdot 3^8 + 3^{12} + O(3^{13})\right) t^4 +$$
$$\left(2\cdot 3^4 + 2\cdot 3^7 + 3^8 + 3^9 + 2\cdot 3^{11} + 2\cdot 3^{12} + 2\cdot 3^{13} + O(3^{14})\right) t^5 + \left(3^4 + 2\cdot 3^5 + 3^6 + 3^7 + 2\cdot 3^8 + 3^9 + 3^{11} + 3^{12} + O(3^{14})\right) t^6 +$$
$$\left(2\cdot 3^6 + 2\cdot 3^7 + 3^8 + 3^{10} + 2\cdot 3^{11} + 3^{12} + 2\cdot 3^{14} + O(3^{16})\right) t^7 + \left(2\cdot 3^8 + 2\cdot 3^9 + 3^{11} + 3^{12} + 2\cdot 3^{14} + 2\cdot 3^{15} + 2\cdot 3^{16} + O(3^{18})\right) t^8 +$$
$$\left(2\cdot 3^6 + 2\cdot 3^9 + 2\cdot 3^{10} + 3^{12} + 2\cdot 3^{14} + 2\cdot 3^{15} + O(3^{16})\right) t^9 + \left(2\cdot 3^9 + 3^{10} + 2\cdot 3^{11} + 3^{13} + 3^{16} + 3^{17} + O(3^{19})\right) t^{10} + \cdots,$$

which just has a simple zero at $t = 0$, corresponding to $(1, 1/2)$.

Repeating this for each residue disk, we find that each residue disk has a simple zero at the rational point and no others, which gives that

$$X(\mathbb{Q}_3)_1 = X(\mathbb{Q})_{\text{known}} = \{\infty, (0, \pm 1/2), (-1, \pm 1/2), (1, \pm 1/2)\},$$

and proves that

$$X(\mathbb{Q}) = \{\infty, (0, \pm 1/2), (-1, \pm 1/2), (1, \pm 1/2)\}.$$

Here is `SageMath` code to carry out this computation:

```
R.<x> = QQ[]
X = HyperellipticCurve(x^5-2*x^3+x+1/4)
p = 3
K = Qp(p,15) #some amount of precision loss
XK = X.change_ring(K)
a,b,_,_ = XK.coleman_integrals_on_basis(XK(0,1/2),XK(-1,-1/2))
for P in [X(0,1,0), X(0,1/2), X(-1,1/2), X(1,1/2)]:
    x,y = X.local_coord(P)
    t = x.parent().gen()
    S = K[[t]]
    dx = x.derivative()
    omega0 = dx/(2*y)
    omega1 = x*omega0
    try:
        I = (b*S(omega0)-a*S(omega1)).integral()(p*t)
    except TypeError:
        I = (b*S(omega0.power_series())-a*S(omega1.power_series())).integral()(p*t)
        I = I.power_series()
    coeffval = min(c.valuation() for c in I.list())
    I = I/p^coeffval
    r = (I).truncate(20).roots()
```

```
    [rt for rt in r if (rt[0]).valuation() > -1]
```

We note that `Magma` has an implementation of a combination of the Chabauty–Coleman method with the Mordell-Weil sieve for genus 2 curves of rank 1.

```
> R<x> := PolynomialRing(Rationals());
> X := HyperellipticCurve(x^5-2*x^3+x+1/4);
> C := IntegralModel(X);
> RationalPoints(C:Bound:=1000);
{@ (1 : 0 : 0), (-1 : -1 : 1), (-1 : 1 : 1), (0 : -1 : 1), (0 : 1 : 1), (1 : -1
: 1), (1 : 1 : 1) @}
> J := Jacobian(C);
> P := J!(C![0,1] - C![-1,-1]); //a point of infinite order
> assert(Order(P)) eq 0;
> Chabauty(P);
{ (1 : 1 : 1), (0 : -1 : 1), (0 : 1 : 1), (1 : 0 : 0), (-1 : 1 : 1), (1 : -1 :
1), (-1 : -1 : 1) }
{ 11, 19, 41, 43, 83, 179, 211 }
[ 2, 7, 23, 3, 13, 3 ]
```

1.4. **More p-adic cohomology.** We saw how Kedlaya's zeta function algorithm played a crucial role in computing Coleman integrals on hyperelliptic curves. It would certainly be useful to compute Coleman integrals on curves beyond those that are hyperelliptic. And indeed, in the years since Kedlaya's algorithm, a number of related zeta function algorithms were given: for superelliptic curves by Gaudry–Gürel [GG01], hyperelliptic curves given by an even degree model by Harrison [Har12], hyperelliptic curves in characteristic 2 by Denef–Vercauteren [DV06b], C_{ab} curves by Denef–Vercauteren [DV06a], and nondegenerate curves by Castryck–Denef–Vercauteren [CDV06]. However, the more general of these algorithms were not obviously practical and were not implemented.

More recently, Tuitman [Tui16, Tui17] gave an efficient algorithm to compute the action of Frobenius on rigid cohomology on smooth curves, by using a plane model with a map to $\mathbb{P}^1$. We give a survey of Tuitman's algorithm and show how it can be turned into an algorithm to compute Coleman integrals on plane curves.

First, we set up Tuitman's algorithm from the point of view of explicit Coleman integration, as done by Balakrishnan–Tuitman [BT20]. Let X be a nice curve over $\mathbb{Q}$ of genus g, birational to

$$Q(x,y) = y^{d_x} + Q_{d_x-1}y^{d_x-1} + \cdots + Q_0 = 0,$$

such that $Q(x,y)$ is irreducible and $Q_i(x) \in \mathbb{Z}[x]$ for $i = 0, \ldots, d_x - 1$. Here is a rough outline of Tuitman's algorithm:

(1) Consider the map: $x : X \to \mathbb{P}^1$ and remove the ramification locus $r(x)$ of x. (This is the analogue of removing the Weierstrass points in Kedlaya's algorithm.)
(2) Choose a lift of Frobenius sending $x \mapsto x^p$ and compute the image of y via Hensel lifting.
(3) Compute the action of Frobenius on differentials and reduce pole orders using relations in cohomology via Lauder's fibration algorithm.

Then for a basis $\{\omega_i\}_{i=0}^{2g-1}$ of $\mathrm{H}^1_{\mathrm{rig}}(X \otimes \mathbb{Q}_p)$, Tuitman's algorithm computes

$$\phi^*\omega_i = dh_i + \sum_{j=0}^{2g-1} M_{ji}\omega_j,$$

and, as before, this algorithm for computing the action of Frobenius on cohomology can be used to give an algorithm for Coleman integration.

We let $\Delta(x) \in \mathbb{Z}[x]$ be the discriminant of Q with respect to y, and let $r(x) = \Delta/\gcd(\Delta, d\Delta/dx)$. Note that $r(x)$ is squarefree and divides $\Delta(x)$.

Set

$$S = \mathbb{Z}_p\langle x, 1/r\rangle, \qquad S^\dagger = \mathbb{Z}_p\langle x, 1/r\rangle^\dagger,$$
$$R = \mathbb{Z}_p\langle x, 1/r, y\rangle/(Q), \qquad R^\dagger = \mathbb{Z}_p\langle x, 1/r, y\rangle^\dagger/(Q),$$

where $\langle\ \rangle^\dagger$ denotes the ring of overconvergent functions given by weak completion of the corresponding polynomial ring.

Definition 1.44. Let $W^0 \in \mathrm{GL}_{d_x}(\mathbb{Q}[x, 1/r])$ and $W^\infty \in \mathrm{GL}_{d_x}(\mathbb{Q}[x, 1/x, 1/r])$ denote matrices such that, if we denote

$$b_j^0 = \sum_{i=0}^{d_x-1} W^0_{i+1,j+1} y^i \quad \text{and} \quad b_j^\infty = \sum_{i=0}^{d_x-1} W^\infty_{i+1,j+1} y^i$$

for all $0 \le j \le d_x - 1$, then

(1) $[b_0^0, \dots, b_{d_x-1}^0]$ is an integral basis for $\mathbb{Q}(X)$ over $\mathbb{Q}[x]$,
(2) $[b_0^\infty, \dots, b_{d_x-1}^\infty]$ is an integral basis for $\mathbb{Q}(X)$ over $\mathbb{Q}[1/x]$,

where $\mathbb{Q}(X)$ denotes the function field of X. Moreover, let $W \in \mathrm{GL}_{d_x}(\mathbb{Q}[x, 1/x])$ denote the change of basis matrix $W = (W^0)^{-1}W^\infty$.

Example 1.45. Let $X/\mathbb{Q}$ be an odd degree monic hyperelliptic curve of genus g given by the plane model

$$Q(x, y) = y^2 - f(x) = 0.$$

We have that

$$r(x) = f(x)$$

and:

$$W^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad W^\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1/x^{g+1} \end{pmatrix}.$$

This means that $b^0 = [1, y]$ and $b^\infty = [1, y/x^{g+1}]$ are integral bases for the function field of X over $\mathbb{Q}[x]$ and $\mathbb{Q}[1/x]$, respectively.

Definition 1.46. We say that the triple (Q, W^0, W^∞) has *good reduction* at a prime number p if the conditions below (taken from [Tui17, Assumption 1]) are satisfied.

Assumption 1 ([Tui17, Assumption 1])**.**

(1) The discriminant of $r(x)$ is contained in $\mathbb{Z}_p^\times$.
(2) If we let $\mathbb{F}_p(x, y)$ be the field of fractions of $\mathbb{F}_p[x, y]/(Q)$, then:
   (a) The reduction modulo p of $[b_0^0, \dots, b_{d_x-1}^0]$ is an integral basis for $\mathbb{F}_p(x, y)$ over $\mathbb{F}_p[x]$.
   (b) The reduction modulo p of $[b_0^\infty, \dots, b_{d_x-1}^\infty]$ is an integral basis for $\mathbb{F}_p(x, y)$ over $\mathbb{F}_p[1/x]$.
(3) $W^0 \in \mathrm{GL}_{d_x}(\mathbb{Z}_p[x, 1/r])$ and $W^\infty \in \mathrm{GL}_{d_x}(\mathbb{Z}_p[x, 1/x, 1/r])$.
(4) Denote:

$$\mathcal{R}^0 = \mathbb{Z}_p[x]b_0^0 \quad + \ldots + \mathbb{Z}_p[x]b_{d_x-1}^0,$$
$$\mathcal{R}^\infty = \mathbb{Z}_p[1/x]b_0^\infty + \ldots + \mathbb{Z}_p[1/x]b_{d_x-1}^\infty.$$

Then the discriminants of the finite $\mathbb{Z}_p$-algebras $\mathcal{R}^0/(r(x))$ and $\mathcal{R}^\infty/(1/x)$ are contained in $\mathbb{Z}_p^\times$.

Definition 1.47. We say that a point of X^{an} is *very infinite* if its x-coordinate is ∞ and *very bad* if it is either very infinite or its x-coordinate is a zero of $r(x)$.

Definition 1.48. We say that a residue disk (as well as any point inside it) is *infinite* or *bad* if it contains a very infinite or a very bad point, respectively. A point or residue disk is called *finite* if it is not infinite and *good* if it is not bad.

We let U denote the complement of the very bad points in X^{an}.

Definition 1.49. Let $\{\omega_i\}_{i=0,\dots,2g-1}$ be p-adically integral 1-forms on U such that

(1) $\omega_0, \dots, \omega_{g-1}$ form a basis for $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)$,
(2) $\omega_0, \dots, \omega_{2g-1}$ form a basis for $\mathrm{H}^1_{\mathrm{rig}}(X \otimes \mathbb{Q}_p)$,
(3) $\mathrm{ord}_P(\omega_i) \geq -1$ for all i at all finite very bad points P,
(4) $\mathrm{ord}_P(\omega_i) \geq -1 + (\mathrm{ord}_0(W) + 1)e_P$ for all i at all very infinite points P.

In [Tui16, Tui17], it is explained how 1-forms satisfying properties (2)–(4) can be computed. Briefly, one computes a basis for $\mathrm{H}^1_{\mathrm{rig}}(U)$ and uses the kernel of a residue map to extract those 1-forms of the second kind, to produce a basis for $\mathrm{H}^1_{\mathrm{rig}}(X \otimes \mathbb{Q}_p)$. The algorithm can be easily adapted so that (1) is satisfied as well, which is the convention we take.

Definition 1.50. The p-th power Frobenius ϕ acts on $\mathrm{H}^1_{\mathrm{rig}}(X \otimes \mathbb{Q}_p)$, so there exist a matrix $M \in M_{2g\times 2g}(\mathbb{Q}_p)$ and functions $h_0, \dots, h_{2g-1} \in R^\dagger \otimes \mathbb{Q}_p$ such that

$$\phi^*(\omega_i) = dh_i + \sum_{j=0}^{2g-1} M_{ji}\omega_j$$

for $i = 0, \dots, 2g-1$.

After we compute the action of Frobenius on a 1-form, we need to reduce the pole order using relations in cohomology. Tuitman's algorithm uses Lauder's fibration algorithm, which solves for a cohomologous differential of lower pole order using a linear system. Tuitman applies it first to points not lying over infinity:

Proposition 1.51. *Let r' denote dr/dx for points not over infinity. For all $\ell \in \mathbb{N}$ and every $w \in \mathbb{Q}_p[x]^{\oplus d_x}$, there exist vectors $u, v \in \mathbb{Q}_p[x]^{\oplus d_x}$ such that* $\deg(v) < \deg(r)$ *and*

$$\frac{\sum_{i=0}^{d_x-1} w_i b_i^0}{r^\ell}\frac{dx}{r} = \left(d\frac{\sum_{i=0}^{d_x-1} v_i b_i^0}{r^\ell}\right) + \frac{\sum_{i=0}^{d_x-1} u_i b_i^0}{r^{\ell-1}}\frac{dx}{r}$$

Proof. Since r is separable, r' is invertible in $\mathbb{Q}_p[x]/r$. We check that there is a unique solution v to the $d_x \times d_x$ linear system

$$(M/r' - \ell I)\, v \equiv w/r' \mod r$$

over $\mathbb{Q}_p[r]/(r)$: take

$$u = \frac{w - (M - \ell r' I)v}{r} - \frac{dv}{dx}.$$

□

For reducing pole orders at points over infinity, we have the following proposition:

Proposition 1.52. *For every vector $w \in \mathbb{Q}_p[x,1/x]^{\oplus d_x}$ with $\mathrm{ord}_\infty(w) \leq -\deg r$, there exist $u, v \in \mathbb{Q}_p[x,1/x]^{\oplus d_x}$ with $\mathrm{ord}_\infty(u) > \mathrm{ord}_\infty(w)$ such that*

$$\left(\sum_{i=0}^{d_x-1} w_i b_i^\infty\right)\frac{dx}{r} = d\left(\sum_{i=0}^{d_x-1} v_i b_i^\infty\right) + \left(\sum_{i=0}^{d_x-1} u_i b_i^\infty\right)\frac{dx}{r}.$$

Here is Tuitman's algorithm for computing the matrix M and the functions $h_0, \dots, h_{2g-1}$:

Algorithm 1.53 (Tuitman's algorithm [Tui16, Tui17])**.**

Input:

- A prime $p > 2$ of good reduction (in the sense of Definition 1.46) for a nice curve $X/\mathbb{Q}$
- A basis $\{\omega_i\}$ of $\mathrm{H}^1_{\mathrm{rig}}(X \otimes \mathbb{Q}_p)$

Output: The matrix $M \in M_{2g\times 2g}(\mathbb{Q}_p)$ and overconvergent functions $h_i \in R^\dagger \otimes \mathbb{Q}_p$ such that $\phi^*\omega_i = dh_i + \sum_{j=0}^{2g-1} M_{ji}\omega_j$.

(1) Compute the Frobenius lift: set $\phi(x) = x^p$ and determine the elements $\phi(1/r) \in S^\dagger$ and $\phi(y) \in R^\dagger$ by Hensel lifting.
(2) Finite pole order reduction: For $i = 0, \dots, 2g-1$, find $h_{i,0} \in R^\dagger \otimes \mathbb{Q}_p$ such that
$$\phi^*(\omega_i) = dh_{i,0} + G_i\left(\frac{dx}{r(x)}\right),$$
where $G_i \in R \otimes \mathbb{Q}_p$ only has poles at very infinite points.
(3) Infinite pole order reduction. For $i = 0, \dots, 2g-1$, find $h_{i,\infty} \in R \otimes \mathbb{Q}_p$ such that
$$\phi^*(\omega_i) = dh_{i,0} + dh_{i,\infty} + H_i\left(\frac{dx}{r(x)}\right),$$
where $H_i \in R \otimes \mathbb{Q}_p$ still only has poles at very infinite points P and satisfies
$$\mathrm{ord}_P(H_i) \geq (\mathrm{ord}_0(W) - \deg(r) + 2)e_P$$
at all these points.
(4) Final reduction: For $i = 0, \dots, 2g-1$, find $h_{i,end} \in R \otimes \mathbb{Q}_p$ such that
$$\phi^*(\omega_i) = dh_{i,0} + dh_{i,\infty} + dh_{i,end} + \sum_{j=0}^{2g-1} M_{ji}\omega_j,$$
where $M \in M_{2g\times 2g}(\mathbb{Q}_p)$ is the matrix of ϕ^* on $\mathrm{H}^1_{\mathrm{rig}}(U \otimes \mathbb{Q}_p)$ with respect to the basis $\{\omega_i\}_{i=0}^{2g-1}$.

The matrix M and the functions

$$h_i := h_{i,0} + h_{i,\infty} + h_{i,end}$$

are exactly what we need from [Tui16, Tui17] to compute Coleman integrals, giving the necessary input into Algorithm 1.54.

Algorithm 1.54 (Coleman integration on a plane curve [BT20])**.**

Input:

- A prime $p > 2$ of good reduction (in the sense of Definition 1.46) for a nice curve $X/\mathbb{Q}$
- Points $P, Q \in X(\mathbb{Q}_p)$ not contained in very bad residue disks
- A 1-form ω of the second kind

Output: The Coleman integral $\int_P^Q \omega$.

(1) Since ω is of the second kind, we may write it as a linear combination of a basis $\{\omega_i\}_{i=0}^{2g-1}$ for $\mathrm{H}^1_{\mathrm{rig}}(X \otimes \mathbb{Q}_p)$ together with an exact form. Use Tuitman's algorithm to write $\omega = dh + \sum_{i=0}^{2g-1} a_i \omega_i$, which allows us to specialize to the case of Coleman integrals of basis differentials.
(2) Compute the action of Frobenius on $\mathrm{H}^1_{\mathrm{rig}}(X \otimes \mathbb{Q}_p)$ using Algorithm 1.53 and store M and $h_0, \dots, h_{2g-1}$.
(3) Compute the integrals $\int_P^{\phi(P)} \omega_i$ and $\int_{\phi(Q)}^{Q} \omega_i$ for $i = 0, \dots, 2g-1$ using local coordinates and tiny integrals.
(4) Compute $h_i(P) - h_i(Q)$ for $i = 0, \dots, 2g-1$ and use the system of equations

$$\sum_{j=0}^{2g-1} (M^{\intercal} - I)_{ij} \left(\int_P^Q \omega_j \right) = h_i(P) - h_i(Q) - \int_P^{\phi(P)} \omega_i - \int_{\phi(Q)}^{Q} \omega_i$$

to solve for all $\int_P^Q \omega_i$.

Remark 1.55. As in the case of integrating from a Weierstrass point on a hyperelliptic curve, to integrate from a very bad point B on a plane curve, split up the integral

$$\int_B^Q \omega_i = \int_B^{B'} \omega_i + \int_{B'}^Q \omega_i$$

for B' a point near the boundary of the residue disk of B, then apply Algorithm 1.54 to compute $\int_{B'}^Q \omega_i$ and compute $\int_B^{B'} \omega_i$ using a tiny integral.

Remark 1.56. For precision estimates in Algorithm 1.54, see [BT20, §4].

Example 1.57. We show how the algorithm above can be used to show that a Jacobian of a non-hyperelliptic genus 55 curve has positive rank (for more details about this example, including timing data, see [BT20, §6.4]).

We consider the genus 55 curve X with plane model given by $Q(x, y) = 0$ below:

$$\begin{aligned} Q(x,y) = {} & x^{11}y - x^7y^5 - x^6y^6 - x^4y^8 + xy^{11} + y^{12} + x^{11} - x^{10}y + x^8y^3 - x^6y^5 + x^5y^6 + x^3y^8 - x^2y^9 - xy^{10} + \\ & y^{11} + x^{10} + x^9y - x^8y^2 + x^7y^3 + x^6y^4 + x^5y^5 - x^4y^6 + xy^9 + y^{10} - x^9 + x^8y + x^7y^2 + x^6y^3 + x^5y^4 + \\ & x^4y^5 + x^3y^6 - x^2y^7 + y^9 + x^8 - x^7y + x^6y^2 - x^5y^3 + xy^7 + y^8 + x^7 + x^6y + x^5y^2 - x^2y^5 - xy^6 + \\ & y^7 - x^6 - x^4y^2 - x^2y^4 + xy^5 - x^5 + x^3y^2 - x^2y^3 + y^5 - x^4 + x^3y + x^2y^2 + xy^3 + y^4 - x^2y - xy^2 + \\ & y^3 - x^2 - xy + x + y. \end{aligned}$$

Let $p = 7$ and consider $P_1 = (0, 0)$ and $P_2 = (1, 0)$, which are each good points on X. We compute the Coleman integrals $\left\{ \int_{P_1}^{P_2} \omega_i \right\}_{i=1}^{110}$ for the basis $\{\omega_i\}$ of $\mathrm{H}^1_{\mathrm{rig}}(X \otimes \mathbb{Q}_p)$ constructed as in Definition 1.49 with $N = 5$ as our precision. We find that

$$\int_{P_1}^{P_2} \omega_1 = 5 \cdot 7 + O(7^2),$$

and thus the Jacobian of X has positive rank.

The `Magma` code for this example is available at `./examples/g55.m` in [BTb].

Project 1.58 (Coleman integration for curves over number fields)**.** Give an algorithm to compute Coleman integrals on curves over number fields and implement the algorithm. To start, see the GitHub repository of Balakrishnan–Tuitman [BTb] for plane curves defined over $\mathbb{Q}$. Before implementing, it would be good to think through the current scope of curves and number fields that are practical.

Project 1.59 (A Chabauty–Coleman solver)**.** Use the project above as well as estimates on precision of p-adic power series to give a Chabauty–Coleman solver for curves over number fields that would take as input a genus g curve X defined over a number field K with $r = \operatorname{rk} J(K) < g$, a prime $\mathfrak{p}$ of good reduction, and r generators of the Mordell–Weil group modulo torsion and output the set $X(K_{\mathfrak{p}})_1$. To start, see the GitHub repositories of Balakrishnan–Tuitman [BTb] and Hashimoto–Morrison [HM].

1.5. **Iterated Coleman integrals.** Let $X/\mathbb{Q}$ be a nice curve of genus g with a plane model and let p be a prime of good reduction. In [Col82], Coleman described a construction of iterated p-adic integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with applications to Beilinson's conjecture. This was extended by Coleman–de Shalit [CdS88] to any curve and by Besser [Bes02] to higher-dimensional varieties. (For the classical theory of iterated integrals, see the work of Chen [Che71].) We will see an application of iterated integrals to studying rational points on curves in §1.6.

By an *iterated Coleman integral* we mean an iterated path integral

$$\int_P^Q \eta_n \dots \eta_1 = \int_0^1 \int_0^{t_1} \dots \int_0^{t_{n-1}} f_n(t_n) \dots f_1(t_1) dt_n \dots dt_1. \tag{12}$$

We will henceforth use notation on the left hand side of (12) to describe iterated integrals, where the implicit integrations are with respect to a dummy variable: e.g.,

$$\int_P^Q \eta_2 \eta_1 := \int_P^Q \eta_2(R) \int_P^R \eta_1 = \int_P^Q \eta_2(R) I(R),$$

where $I(R) = \int_P^R \eta_1$.

The main idea is to apply an algorithm for computing the action of Frobenius on p-adic cohomology (e.g., Kedlaya or Tuitman) to produce the relationship

$$\phi^* \omega_i = dh_i + \sum_{j=0}^{2g-1} M_{ji} \omega_j,$$

observe that the eigenvalues of $M^{\otimes n}$, are not 1, and reduce the computation of an n-fold iterated integral to a computation of an $(n-1)$-fold iterated integral. For instance, in writing down a linear system for computing single Coleman integrals, we used the fundamental theorem of calculus to produce the constants

$$\int_P^Q dh_i = h_i(Q) - h_i(P),$$

and now we will apply this idea inductively. As before, iterated integrals between points in the same residue disk can be computed using a local coordinate at one point and (iteratively) integrating power series.

More formally, here is how we compute a tiny iterated integral:

Algorithm 1.60 (Tiny iterated integral on a plane curve X)**.**

Input:

- A prime $p > 2$ of good reduction (in the sense of Definition 1.46) for a plane curve $X/\mathbb{Q}$
- Points $P, Q \in X(\mathbb{Q}_p)$ in same residue disk.

Output: The tiny iterated integral $\int_P^Q \eta_1 \dots \eta_n$

(1) Compute a local coordinate $(x(t), y(t))$ at P.
(2) For each k, write $\eta_k(x, y)$ as $\eta_k(t)dt$.

(3) Let $I_{n+1} = 1$. Compute for $k = n, n-1, \dots, 2$
$$I_k = \int_P^{R_{k-1}} \eta_k I_{k+1} = \int_P^{t(R_{k-1})} \eta_k(t) I_{k+1} dt$$
where $t(R_{k-1})$ is parametrizing points in the residue disk of P.
(4) $\int_P^Q \eta_1 \dots \eta_n = \int_P^{t(Q)} \eta_1(t) I_2(t) dt$.

To compute more general iterated Coleman integrals, we will use the following properties.

Proposition 1.61. *Let $\omega_{i_1}, \dots \omega_{i_n}$ be forms of the second kind, holomorphic at $P, Q \in X(\mathbb{Q}_p)$.*

(1) $\displaystyle\int_P^P \omega_{i_1} \dots \omega_{i_n} = 0$

(2) $\displaystyle\sum_{\text{all permutations } \sigma} \int_P^Q \omega_{\sigma(i_1)} \dots \omega_{\sigma(i_n)} = \prod_{j=1}^n \int_P^Q \omega_{i_j}$

(3) $\displaystyle\int_P^Q \omega_{i_1} \dots \omega_{i_n} = (-1)^n \int_Q^P \omega_{i_n} \dots \omega_{i_1}$

As a corollary, we have

Corollary 1.62. $\displaystyle\int_P^Q \underbrace{\omega_i \dots \omega_i}_{n} = \frac{1}{n!} \left(\int_P^Q \omega_i \right)^n$

The following lemma gives the analogue of additivity in endpoints:

Lemma 1.63. *Let $P, P', Q \in X(\mathbb{Q}_p)$. Then*
$$\int_P^Q \omega_{i_1} \dots \omega_{i_n} = \sum_{j=0}^n \int_{P'}^Q \omega_{i_1} \dots \omega_{i_j} \int_P^{P'} \omega_{i_{j+1}} \dots \omega_{i_n}$$

Now for ease of exposition, we will focus our attention on the case of $n = 2$, the double Coleman integrals [Bal13, Bal15].

Applying Lemma 1.63 twice, we may link double integrals between different residue disks:
$$\int_P^Q \omega_i \omega_k = \int_P^{P'} \omega_i \omega_k + \int_{P'}^{Q'} \omega_i \omega_k + \int_{Q'}^Q \omega_i \omega_k + \int_P^{P'} \omega_k \int_{P'}^Q \omega_i + \int_{P'}^{Q'} \omega_k \int_{Q'}^Q \omega_i.$$

We can directly compute double integrals using a linear system. Indeed, using Lemma 1.63, we take $\phi(P)$ and $\phi(Q)$ to be the points in the disks of P and Q, respectively, which gives

$$\int_P^Q \omega_i \omega_k = \int_P^{\phi(P)} \omega_i \omega_k + \int_{\phi(P)}^{\phi(Q)} \omega_i \omega_k + \int_{\phi(Q)}^Q \omega_i \omega_k + \int_P^{\phi(P)} \omega_k \int_{\phi(P)}^Q \omega_i + \int_{\phi(P)}^{\phi(Q)} \omega_k \int_{\phi(Q)}^Q \omega_i. \tag{13}$$

Then we expand the following

$$\begin{aligned} \int_{\phi(P)}^{\phi(Q)} \omega_i \omega_k = \int_P^Q \phi^*(\omega_i \omega_k) &= \int_P^Q \phi^*(\omega_i) \phi^*(\omega_k) \\ &= \int_P^Q (df_i + \sum_{j=0}^{2g-1} M_{ij}^{\intercal} \omega_j)(df_k + \sum_{j=0}^{2g-1} M_{kj}^{\intercal} \omega_j) \\ &= c_{ik} + \int_P^Q \left(\sum_{j=0}^{2g-1} M_{ij}^{\intercal} \omega_j \right) \left(\sum_{j=0}^{2g-1} M_{kj}^{\intercal} \omega_j \right), \end{aligned} \tag{14}$$

where

$$c_{ik} = \int_P^Q df_i(R)(f_k(R)) - f_k(P)(f_i(Q) - f_i(P)) + \int_P^Q \sum_{j=0}^{2g-1} M_{ij}^{\intercal}\omega_j(R)(f_k(R) - f_k(P))$$
$$+ f_i(Q)\int_P^Q \sum_{j=0}^{2g-1} M_{kj}^{\intercal}\omega_j - \int_P^Q f_i(R)(\sum_{j=0}^{2g-1} M_{kj}^{\intercal}\omega_j(R)).$$

Putting together (13) and (14), we get

$$\begin{pmatrix} \vdots \\ \int_P^Q \omega_i\omega_k \\ \vdots \end{pmatrix} = (I_{4g^2\times 4g^2} - (M^{\intercal})^{\otimes 2})^{-1} \begin{pmatrix} \vdots \\ c_{ik} - \int_{\phi(P)}^P \omega_i\omega_k - \left(\int_P^Q \omega_i\right)\left(\int_{\phi(P)}^P \omega_k\right) \\ - \left(\int_Q^{\phi(Q)} \omega_i\right)\left(\int_{\phi(P)}^{\phi(Q)} \omega_k\right) + \int_{\phi(Q)}^Q \omega_i\omega_k \\ \vdots \end{pmatrix}.$$

Algorithm 1.64 (Double Coleman integrals [Bal13, Bal15])**.**
Input:

- A prime $p > 2$ of good reduction (in the sense of Definition 1.46) for a plane curve $X/\mathbb{Q}$
- Points $P, Q \in X(\mathbb{Q}_p)$ in the region of overconvergence for the lift of p-power Frobenius

Output: The double integrals $\left(\int_P^Q \omega_i\omega_j\right)_{i,j=0}^{2g-1}$.

(1) Use Algorithm 1.54 to compute the single integrals $\int_P^Q \omega_i, \int_{\phi(P)}^{\phi(Q)} \omega_i$ for all i.
(2) Use Algorithm 1.60 to compute $\int_{\phi(P)}^P \omega_i\omega_k, \int_{\phi(Q)}^Q \omega_i\omega_k$ for all i, k
(3) Compute the constants c_{ik} for all i, k using single integrals.
(4) Recover the double integrals using the linear system

$$\begin{pmatrix} \vdots \\ \int_P^Q \omega_i\omega_k \\ \vdots \end{pmatrix} = (I_{4g^2\times 4g^2} - (M^{\intercal})^{\otimes 2})^{-1} \begin{pmatrix} \vdots \\ c_{ik} - \int_{\phi(P)}^P \omega_i\omega_k - \left(\int_P^Q \omega_i\right)\left(\int_{\phi(P)}^P \omega_k\right) \\ - \left(\int_Q^{\phi(Q)} \omega_i\right)\left(\int_{\phi(P)}^{\phi(Q)} \omega_k\right) + \int_{\phi(Q)}^Q \omega_i\omega_k \\ \vdots \end{pmatrix}.$$

Remark 1.65. In [Bal13, Bal15], Algorithm 1.64 was described and implemented for hyperelliptic curves, as it used Kedlaya's algorithm (and Harrison's generalization) for the Frobenius step. With Tuitman's algorithm in place of Kedlaya's, one can run the algorithm for (a plane model of) a nice curve.

Example 1.66. Let $X/\mathbb{Q}$ be the genus 2 curve

$$y^2 = x^5 - 2x^4 + 2x^3 - x + 1$$

which is [LMF20a] and has good reduction at $p = 5$.

Using Algorithm 1.64, we compute 5-adic double Coleman integrals between the points $P = (0, 1)$ and $Q = (1, 1)$, where $\omega_i = \frac{x^i}{2y}dx$:

$$\begin{pmatrix} \int_P^Q \omega_0\omega_0 \\ \int_P^Q \omega_0\omega_1 \\ \int_P^Q \omega_0\omega_2 \\ \int_P^Q \omega_0\omega_3 \\ \int_P^Q \omega_1\omega_0 \\ \int_P^Q \omega_1\omega_1 \\ \int_P^Q \omega_1\omega_2 \\ \int_P^Q \omega_1\omega_3 \\ \int_P^Q \omega_2\omega_0 \\ \int_P^Q \omega_2\omega_1 \\ \int_P^Q \omega_2\omega_2 \\ \int_P^Q \omega_2\omega_3 \\ \int_P^Q \omega_3\omega_0 \\ \int_P^Q \omega_3\omega_1 \\ \int_P^Q \omega_3\omega_2 \\ \int_P^Q \omega_3\omega_3 \end{pmatrix} = \begin{pmatrix} 3\cdot 5^2 + 3\cdot 5^3 + 5^4 + 4\cdot 5^5 + 5^6 + O(5^8) \\ 3\cdot 5^2 + 3\cdot 5^3 + 2\cdot 5^4 + 2\cdot 5^5 + 5^6 + 3\cdot 5^7 + O(5^8) \\ 2\cdot 5 + 4\cdot 5^2 + 5^3 + 3\cdot 5^4 + 4\cdot 5^5 + O(5^7) \\ 4\cdot 5 + 5^2 + 4\cdot 5^3 + 5^4 + 3\cdot 5^5 + 2\cdot 5^6 + O(5^7) \\ 5^3 + 3\cdot 5^4 + 2\cdot 5^5 + 4\cdot 5^6 + 5^7 + O(5^8) \\ 2\cdot 5^2 + 5^3 + 4\cdot 5^4 + 5^6 + O(5^8) \\ 2\cdot 5^2 + 4\cdot 5^3 + 5^4 + 3\cdot 5^5 + 5^6 + O(5^7) \\ 1 + 4\cdot 5 + 2\cdot 5^2 + 5^3 + 5^4 + 4\cdot 5^5 + 3\cdot 5^6 + O(5^7) \\ 4\cdot 5^3 + 3\cdot 5^4 + 3\cdot 5^5 + 3\cdot 5^6 + O(5^7) \\ 5 + 2\cdot 5^2 + 4\cdot 5^3 + 4\cdot 5^4 + 5^5 + O(5^7) \\ 2 + 4\cdot 5 + 4\cdot 5^2 + 5^3 + 2\cdot 5^4 + 4\cdot 5^5 + O(5^6) \\ 3 + 2\cdot 5 + 2\cdot 5^2 + 4\cdot 5^3 + 3\cdot 5^4 + 4\cdot 5^5 + O(5^6) \\ 4\cdot 5 + 5^2 + 3\cdot 5^3 + 4\cdot 5^4 + 3\cdot 5^5 + 3\cdot 5^6 + O(5^7) \\ 4 + 4\cdot 5 + 5^2 + 4\cdot 5^3 + 3\cdot 5^5 + 4\cdot 5^6 + O(5^7) \\ 3 + 4\cdot 5 + 4\cdot 5^2 + 2\cdot 5^5 + O(5^6) \\ 2 + 3\cdot 5 + 4\cdot 5^2 + 5^4 + 4\cdot 5^5 + O(5^6) \end{pmatrix}.$$

Using `SageMath` code available on GitHub [Bal], here is how to generate the values of the double integrals above:

```
R.<x> = QQ[]
X = HyperellipticCurve(x^5-2*x^4+2*x^3-x+1)
K = Qp(5,8)
XK = X.change_ring(K)
P = XK(0,1)
Q = XK(1,1)
XK.double_integrals_on_basis(P,Q)
```

Project 1.67. There is a certain amount of redundancy that allows one to express double (or higher iterated) integrals in terms of single integrals. For instance, looking at double integrals in the case of $g = 1$, we have that $\int_P^Q \omega_i\omega_i = \frac{1}{2}\left(\int_P^Q \omega_i\right)^2$ for $i = 0, 1$ and $\int_P^Q \omega_0\omega_1 + \int_P^Q \omega_1\omega_0 = \int_P^Q \omega_0 \int_P^Q \omega_1$. Can these relations be used to give a more efficient algorithm to compute double (or higher iterated) integrals?

1.6. **An application (preview).** Let $\mathcal{E}/\mathbb{Z}$ be the minimal regular model of an elliptic curve. Let $\mathcal{X} = \mathcal{E} \setminus \mathcal{O}$. Let $\omega_0 = \frac{dx}{2y+a_1x+a_3}$, $\omega_1 = x\omega_0$ in Weierstrass coordinates.

Let b be a tangential basepoint at the point at infinity or an integral 2-torsion point. (For more about tangential basepoints, see Deligne [Del89] or Besser [Bes12, §1.5.4]. Roughly, the issue is that ω_1 has a pole at the point at infinity, so to make sense of an integral from the point at infinity, we must normalize with respect to a choice of tangent vector, which essentially means that we are fixing a direction at b.) Let p be a prime of good reduction. Suppose $\mathcal{E}$ has analytic rank 1 and Tamagawa product 1. Consider

$$\log(z) = \int_b^z \omega_0, \qquad D_2(z) = \int_b^z \omega_0\omega_1. \tag{15}$$

One can think of $\log(z)$ as the Coleman integral extending log on the formal group of $\mathcal{E}/\mathbb{Z}_p$. The function D_2 is labeled as such to suggest a dilogarithm.

Theorem 1.68 ([Kim10, BKK11]). *Suppose P is a point of infinite order in $\mathcal{E}(\mathbb{Z})$. Then $\mathcal{X}(\mathbb{Z}) \subseteq \mathcal{E}(\mathbb{Z})$ is in the zero set of*

$$f(z) = (\log(P))^2 D_2(z) - (\log(z))^2 D_2(P),$$

or in other words, $\frac{D_2(z)}{(\log(z))^2}$ is constant on integral points.

We will return to this result and discuss how it is related to p-adic height pairings in the following section.

2. p-adic heights on Jacobians of curves

From a computational point of view, the main idea of quadratic Chabauty is to replace the *linear* relations that make it possible to cut out rational points among p-adic points in the method of Chabauty–Coleman by *bilinear* relations. This can be achieved using the theory of p-adic heights, developed in various degrees of generality by Bernardi [Ber81], Néron [Nér76], Perrin-Riou [PR83], Schneider [Sch82], Mazur–Tate [MT83], Zarhin [Zar90], Iovita–Werner [IW03], Coleman–Gross [CG89], and Nekovář [Nek93].

Most of these constructions are quite similar to constructions of the real valued (or Néron-Tate) height pairing. Recall that this is a symmetric bilinear pairing $A(K) \times A(K) \to \mathbb{R}$, where A is an abelian variety over a global field K, such that the associated quadratic form $\hat{h}\colon A(K) \to \mathbb{R}$ satisfies the Northcott property: for all real numbers B the set of points $P \in A(K)$ such that $\hat{h}(P) < B$ is finite. The latter property has no analogue in the p-adic world, but the bilinearity carries over.

2.1. p-adic heights on elliptic curves.

We begin with a discussion of p-adic heights on elliptic curves defined over the rationals, following the work of Mazur–Stein–Tate [MST06]. In this context already, we can see a hint of some objects that will show up in explicit quadratic Chabauty.

Let $E/\mathbb{Q}$ be an elliptic curve, given by a Weierstrass equation with integral coefficients, and let $\mathcal{O}$ be the point at infinity. Let $p \geq 5$ be a prime of good ordinary reduction for E. Let $P \in E(\mathbb{Q})$ be a nonzero point. Write

$$P = (x(P), y(P)) = \left(\frac{a(P)}{d(P)^2}, \frac{b(P)}{d(P)^3}\right),$$

where

$$a(P), b(P), d(P) \in \mathbb{Z}, \quad d(P) \geq 1, \quad \gcd(a(P), d(P)) = 1 = \gcd(b(P), d(P)).$$

We call $d(P)$ the *denominator* of P. Suppose that P satisfies two conditions:

(i) P reduces to $\mathcal{O}$ in $E(\mathbb{F}_p)$;
(ii) P reduces to a nonsingular point of $E(\mathbb{F}_\ell)$ for all bad primes ℓ.

Fix a branch $\log_p\colon \mathbb{Q}_p^* \to \mathbb{Q}_p$ of the p-adic logarithm.

Definition 2.1. The *cyclotomic p-adic height* on such a point $P \in E(\mathbb{Q})$ is

$$h(P) = \frac{1}{p} \log_p\left(\frac{\sigma(P)}{d(P)}\right) \in \mathbb{Q}_p,$$

where $\sigma(P)$ is the p-adic sigma function associated to $E/\mathbb{Z}_p$, characterized in Theorem 2.3 below.

Remark 2.2. More generally, p-adic heights depend on a choice of idèle class character, see Remark 2.15. Over $\mathbb{Q}$, up to scalars, this is uniquely determined and is the *cyclotomic* character.

Mazur and Tate gave 11 different characterizations of the p-adic sigma function [MT91]. We will describe one characterization, which is particularly useful for computations.

Let $x(t) = t^{-2} + \cdots \in \mathbb{Z}_p((t))$ be x in the formal group of $E/\mathbb{Z}_p$; then $y(t) = t^{-3} + \cdots \in \mathbb{Z}_p((t))$.

Theorem 2.3 (Mazur–Tate [MT91])**.** *There is exactly one odd*[7]

$$\sigma(t) = t + \cdots \in t\,\mathbb{Z}[\![t]\!].$$

and constant $c \in \mathbb{Z}_p$ that together satisfy the p-adic differential equation:

$$x(t) + c = -\frac{d}{\omega}\left(\frac{1}{\sigma}\frac{d\sigma}{\omega}\right),$$

where ω is the invariant differential associated to the chosen Weierstrass model for E

$$\omega = \frac{dx}{2y + a_1x + a_3}, \quad \text{and } c = \frac{a_1^2 + 4a_2 - \mathbf{E}_2(E,\omega)}{12}.$$

Remark 2.4. Indeed, one can define other p-adic sigma functions, depending on the choice of constant c. These other constants result in other p-adic heights; for instance, taking $c = 0$ gives the Bernardi p-adic height [Ber81]. One can also drop the assumption of ordinarity in this context.

We will return to $\mathbf{E}_2(E,\omega)$ in a bit.

Lemma 2.5. *The height function h extends uniquely to the full Mordell–Weil group $E(\mathbb{Q})$ so that $h(nP) = n^2h(P)$ for all $n \in \mathbb{Z}$ and $P \in E(\mathbb{Q})$. For $P, Q \in E(\mathbb{Q})$ setting*

$$(P,Q) = h(P) + h(Q) - h(P+Q),$$

we get a symmetric bilinear pairing on $E(\mathbb{Q})$.

To compute $h(Q)$ for arbitrary $Q \in E(\mathbb{Q}) \setminus \{\mathcal{O}\}$, let $n_1 = \#E(\mathbb{F}_p)$ and $n_2 = \operatorname{lcm}(\{c_v\})$, where the c_v are the Tamagawa numbers. Let $n = \operatorname{lcm}(n_1, n_2)$. Then $P := nQ$ satisfies (i) and (ii) needed earlier, so we may compute $h(P) = h(nQ)$, and then

$$h(Q) = \frac{1}{n^2}h(nQ) = \frac{1}{n^2}h(P).$$

One reason why the p-adic height is interesting is that, in analogy with canonical height, one can define the p-adic regulator Reg_p of $E/\mathbb{Q}$ as the determinant of the matrix of pairings on $E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}}$. Then the p-adic regulator fits into a p-adic Birch and Swinnerton-Dyer conjecture. The simplest instance of this is as follows:

Conjecture 2.6 (Mazur–Tate–Teitelbaum [MTT86])**.** *Suppose E has good ordinary reduction at p. Let $\mathcal{L}_p(E,T)$ be the p-adic L-function attached to $E/\mathbb{Q}$. Then we have*

(1)

$$\operatorname{ord}_{T=0}\mathcal{L}_p(E,T) = \operatorname{rk}E(\mathbb{Q})$$

(2) The leading coefficient $\mathcal{L}_p^*(E,0)$ of the expansion of the p-adic L-function at $T = 0$ satisfies the following:

$$\mathcal{L}_p^*(E,0) = \frac{\epsilon_p \prod_v c_v |Ш(E/\mathbb{Q})| \operatorname{Reg}_p}{(\#E(\mathbb{Q})_{\text{tors}})^2}$$

where $\epsilon_p = (1-\alpha^{-1})^2$ is the p-adic multiplier and α is the unit root of $x^2 - a_px + p = 0$.

[7] By odd, we mean that $\sigma(I(t)) = -\sigma(t)$ where $I(t) = -t - a_1t^2 + \cdots$ is the formal inverse law.

Remark 2.7. For numerical methods for the computation of the quantities appearing in the conjecture and applications, see the work of Stein–Wuthrich [SW13].

Remark 2.8. Unlike the classical L-function, where one considers the analytic rank to be the order of vanishing at $s = 1$, for the p-adic L-function one considers the p-adic analytic rank to be the order of vanishing at $T = 0$. The series expansion of the p-adic L-function is computed with respect to a choice of topological generator of the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. For further details, see [SW13, §3].

Remark 2.9. Note that for the conjecture to hold as stated, one must choose a different normalization of the p-adic height than that given in Definition 2.1: instead one takes $h(P) = 2\log_p\left(\frac{\sigma(P)}{d(P)}\right)$.

Remark 2.10. For a precision analysis of the computation of p-adic heights on elliptic curves, see the work of Harvey [Har08, Theorem 3].

Example 2.11. Both `SageMath` and `Magma` have implementations of p-adic heights on elliptic curves over $\mathbb{Q}$ for good ordinary primes $p \geq 5$ (with `SageMath` more generally handling semistable reduction). Beware that the normalizations[8] may be slightly different! In `SageMath`, the normalization[9] is chosen for the p-adic Birch–Swinnerton-Dyer conjecture to hold as stated in [MTT86], so differs from [MST06] by a factor of $2p$:

```
sage: E = EllipticCurve([1,1])
sage: p = 5
sage: h = E.padic_height(p,8)
sage: P = E(0,1)
sage: for i in range(1,4):
....:     1/i^2*h(i*P)
....:
2*5 + 4*5^3 + 4*5^4 + 5^6 + 5^7 + O(5^8)
2*5 + 4*5^3 + 4*5^4 + 5^6 + 5^7 + O(5^8)
2*5 + 4*5^3 + 4*5^4 + 5^6 + 5^7 + O(5^8)
```

`Magma`'s normalization[10] is that of [Har08] and is $2p$ (or $-2p$ in some cases) times that in other papers. In this example, it differs from `SageMath` by a sign:

```
> E:=EllipticCurve([1,1]);
> P:=E![0,1];
> pAdicHeight(P,5);
1480998027523*5 + O(5^20)
```

Using the Northcott property, it is easy to see that the canonical height of a point $P \in E(\mathbb{Q})$ vanishes if and only if P is torsion. Similarly, we have

Conjecture 2.12 (Schneider [Sch82])**.** *The cyclotomic p-adic height pairing is nondegenerate. Equivalently,* Reg_p *is nonzero.*

For elliptic curves with complex multiplication, Bertrand [Ber75] proved using p-adic transcendence theory that the p-adic height of a non-torsion point is nonzero, which proves Schneider's conjecture if the curve has rank 1, but this is still all we know.

[8] This is something to be aware of regarding the literature on heights as well.

[9] http://doc.sagemath.org/html/en/reference/curves/sage/schemes/elliptic_curves/ell_rational_field.html

[10] https://magma.maths.usyd.edu.au/magma/handbook/text/1485#16955

Remark 2.13. It is also of interest to study the p-adic height in families of elliptic curves, as initiated by Wuthrich [Wut04], who used this to derive interesting results in view of Schneider's conjecture. Recently, Bianchi [Bia19] gave an algorithm using p-adic cohomology to compute p-adic heights in families of elliptic curves.

To complete our construction of the p-adic height, we now discuss how to compute the special value $\mathbf{E}_2(E,\omega)$. Katz [Kat73, App. 2] gives an interpretation to $\mathbf{E}_2(E,\omega)$ as the "direction" of the *unit root subspace* W of Frobenius acting on Monsky–Washnitzer cohomology for E. (Recall that in the case of ordinary reduction, the characteristic polynomial of Frobenius has a distinct unit root. The eigenspace corresponding to the unit root is the so-called unit root subspace.)

Fix an affine model for $E/\mathbb{Z}_p$ of the form $y^2 = f(x)$ and let ϕ^* be the usual lift of p-power Frobenius from the residue field acting with respect to the basis of $\mathrm{H}^1_{\mathrm{MW}}(E')^-$ (as in §1.3) given by $\left\{\frac{dx}{y}, \frac{xdx}{y}\right\}$.

Let

$$(\phi^*)^n\left(x\frac{dx}{y}\right) = a_n\frac{dx}{y} + b_n\frac{xdx}{y}.$$

Then we have

$$\mathbf{E}_2(E,\omega) \equiv \frac{-12a_n}{b_n} \pmod{p^n}.$$

What does this have to do with integral points on E? Here is a rough idea. We fix an affine minimal model for $E/\mathbb{Z}_p$ of the form $y^2 = f(x)$ and recall the p-adic differential equation satisfied by the p-adic sigma function:

$$x + c = -\frac{d}{\omega}\left(\frac{1}{\sigma}\frac{d\sigma}{\omega}\right),$$

in the formal group of $E/\mathbb{Z}_p$. Rewriting, we have

$$\omega(x+c) = -d\left(\frac{1}{\sigma}\frac{d\sigma}{\omega}\right),$$

and letting $\omega_0 := \omega$ and $\omega_1 := x\omega$, this implies that

$$\int(\omega_1 + c\omega_0) = -\frac{d\sigma}{\sigma\omega_0},$$
$$\omega_0\int(\omega_1 + c\omega_0) = -\frac{d\sigma}{\sigma} = -d\log\sigma,$$
$$\int(\omega_0\omega_1 + c\omega_0\omega_0) = -\log\sigma.$$

Recall from (15) that we have

$$\int\omega_0\omega_1 = D_2$$

and

$$c\int\omega_0\omega_0 = \frac{c}{2}\left(\int\omega_0\right)^2 = \frac{c}{2}(\log)^2,$$

it follows that

$$D_2 + \frac{c}{2}(\log)^2 = -\log\sigma. \tag{16}$$

Now suppose we may interpret the left hand side of (16) as a *Coleman function* [Bes02, Bes12]. We will more formally discuss Coleman functions in Chapter 5 (see §5.2), but for now, we will think of them as solutions to certain p-adic differential equations, analytically continued via Frobenius.

Note that the right hand side of (16) is essentially the global p-adic height without a denominator contribution. Then in the case of a rank 1 elliptic curve, if we are able to impose hypotheses (say we restrict to considering integral points and curves with Tamagawa product 1) under which the denominator does not contribute, we have that the right hand side is further equal to $\alpha(\log)^2$ for some computable constant α. Thus we have that

$$\frac{D_2}{(\log)^2}$$

is constant, which would give Theorem 1.68. Of course, one needs to be more careful at various points of this sketch, but this is essentially our first approach toward a fragment of the quadratic Chabauty method (for integral points on rank 1 elliptic curves), as given by Balakrishnan–Besser [BB15]. To say more, we introduce p-adic heights on Jacobians of curves.

Remark 2.14. Blakestad has constructed p-adic sigma functions for the Jacobian $J/\mathbb{Q}_p$ of a genus 2 curve when J has good reduction, see [Bla18]. His construction is used in forthcoming work by Bianchi to construct a p-adic height pairing on J similar to h.

2.2. **p-adic heights on Jacobians of curves.** Let $X/\mathbb{Q}$ be a nice curve with genus $g \geq 1$, and let p be a prime of good reduction for X. As above, we fix a branch $\log_p\colon \mathbb{Q}_p^* \to \mathbb{Q}_p$. We also fix the following data:

(a) a continuous nontrivial idèle class character $\chi\colon \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^* \to \mathbb{Q}_p$ (see Remark 2.15 below),
(b) a splitting s of the Hodge filtration (see Remark 2.17 below) on $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_p)$ such that $\ker(s)$ is isotropic with respect to the cup product pairing.

Remark 2.15. Here we mention briefly the role of idèle class characters. More generally, suppose X is a nice curve defined over a number field K. An *idèle class character*

$$\chi = \sum_v \chi_v : \mathbb{A}_K^*/K^* \to \mathbb{Q}_p$$

is a homomorphism that decomposes as a sum of local characters χ_v. Below are some properties of continuous idèle class characters:

- For any prime $\mathfrak{q} \nmid p$ we have $\chi_{\mathfrak{q}}(\mathcal{O}_{K_{\mathfrak{q}}}^*) = 0$ because of continuity. So if $\pi_{\mathfrak{q}}$ is a uniformizer in $K_{\mathfrak{q}}$, then $\chi_{\mathfrak{q}}$ is completely determined by $\chi_{\mathfrak{q}}(\pi_{\mathfrak{q}})$.
- For any $\mathfrak{p} \mid p$, there is a $\mathbb{Q}_p$-linear map $t_{\mathfrak{p}}^{\chi}$ such that we can decompose

(17)
$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{p}}^* & \xrightarrow{\chi_{\mathfrak{p}}} & \mathbb{Q}_p\,, \\ & {\scriptstyle \log_{\mathfrak{p}}} \searrow \quad \nearrow {\scriptstyle t_{\mathfrak{p}}^{\chi}} & \\ & K_{\mathfrak{p}} & \end{array}$$

because $\chi_{\mathfrak{p}}$ takes values in the torsion-free group $(\mathbb{Q}_p, +)$.

If a continuous idèle class character χ is ramified at $\mathfrak{p}$, that is, if the local character $\chi_{\mathfrak{p}}$ does not vanish on $\mathcal{O}_{\mathfrak{p}}^*$, then we can extend $\log_{\mathfrak{p}}$ to

$$\log_{\mathfrak{p}}\colon K_{\mathfrak{p}}^* \to K_{\mathfrak{p}}$$

in such a way that the diagram (17) remains commutative.

Remark 2.16. Working over $K = \mathbb{Q}$, we take χ to be the cyclotomic character, and so $\log_p$ must be the Iwasawa branch of log (that is, with $\log_p(p) = 0$).

Remark 2.17. To fix a splitting of the Hodge filtration on $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_p)$ means to fix a subspace $W := \ker(s)$ of $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_p)$ complementary to the space of holomorphic forms $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)$, i.e.

$$\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_p) = \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1) \oplus W.$$

The isotropy condition on W is equivalent to obtaining a symmetric height pairing in Definition 2.21 and Definition 2.18 below, see [CG89, Proposition 5.2].

In this subsection, we will construct a height pairing h on the Jacobian J of X. Everything can be generalized to number fields K [BBBM21], making choices as above.

Definition 2.18. (Coleman–Gross [CG89]) The *(cyclotomic) p-adic height pairing* is a symmetric bi-additive pairing

$$\mathrm{Div}^0(X) \times \mathrm{Div}^0(X) \to \mathbb{Q}_p,\ (D_1, D_2) \mapsto h(D_1, D_2),$$

for $D_1, D_2 \in \mathrm{Div}^0(X)$ with disjoint support, such that the following holds:

(i) We have

$$\begin{aligned} h(D_1, D_2) &= \sum_{\text{finite primes } v} h_v(D_1, D_2) \\ &= h_p(D_1, D_2) + \sum_{\ell \neq p} h_\ell(D_1, D_2) \\ &= \int_{D_2} \omega_{D_1} + \sum_{\ell \neq p} m_\ell \log_p \ell, \end{aligned}$$

where the integral is a Coleman integral, the sum is finite and $m_\ell \in \mathbb{Q}$ is an intersection multiplicity.

(ii) For $\beta \in \mathbb{Q}(X)^*$, we have

$$h(D, \mathrm{div}(\beta)) = 0.$$

By (ii), h defines a symmetric bilinear pairing $J(\mathbb{Q}) \times J(\mathbb{Q}) \to \mathbb{Q}_p$.

This construction of the p-adic height is similar to the Arakelov-theoretic description of the canonical height due to Faltings and Hriljac.

2.2.1. *Local heights at p.* We will now provide more detail on the local height pairings h_v, beginning with the case $v = p$, as described by Coleman–Gross [CG89] and computed by Balakrishnan–Besser in the case of hyperelliptic curves [BB12, BB21].

We first discuss the construction of the differential ω_{D_1} in (i). Let $\{\omega_0, \ldots, \omega_{2g-1}\}$ be a basis for $\mathrm{H}^1_{\mathrm{dR}}(X)$ with $\{\omega_0, \ldots, \omega_{g-1}\} \in \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)$. Fix a lift ϕ of Frobenius.

Let $T(\mathbb{Q}_p)$ be the group of differentials of the third kind on X. In this section, we take this to mean something stronger than in the previous section: that they have at most simple poles and *integer* residues.

We have a residue divisor homomorphism

$$\mathrm{Res} : T(\mathbb{Q}_p) \to \mathrm{Div}^0(X)\,,\ \omega \mapsto \mathrm{Res}(\omega) = \sum_P (\mathrm{Res}_P\, \omega) P,$$

which induces a short exact sequence

$$0 \to \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1) \to T(\mathbb{Q}_p) \xrightarrow{\mathrm{Res}} \mathrm{Div}^0(X) \to 0. \tag{18}$$

The differential ω_{D_1} will be a differential of the third kind with $\mathrm{Res}(\omega_{D_1}) = D_1$. Here is an example:

Example 2.19. Suppose that X is a hyperelliptic curve with affine model $y^2 = f(x)$, and D_1 is the divisor $(P)-(Q)$ with non-Weierstrass affine points $P, Q \in X(\mathbb{Q})$. We want to write down a differential ω having simple poles with residues $+1$ at P and -1 at Q respectively, and no other poles. For example,

$$\omega = \frac{dx}{2y}\left(\frac{y+y(P)}{x-x(P)} - \frac{y+y(Q)}{x-x(Q)}\right) \tag{19}$$

has residue divisor equal to D_1, as desired. However, adding any holomorphic differential η to ω, and taking the residue divisor map of $\eta+\omega$ will again give us D_1, as we can see by (18). So we must make some choice, which we do below.

We can fix a normalized differential with a given residue divisor using the chosen complementary subspace W. For this, let $T_l(\mathbb{Q}_p)$ denote the group of logarithmic differentials $\frac{df}{f}$ with $f \in \mathbb{Q}_p(X)^*$. Since

$$T_l(\mathbb{Q}_p) \cap \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1) = 0$$

and $\mathrm{Res}\frac{df}{f} = \mathrm{div} f$, from the short exact sequence (18) we get a new short exact sequence

$$0 \to \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1) \to T(\mathbb{Q}_p)/T_l(\mathbb{Q}_p) \to J(\mathbb{Q}_p) \to 0.$$

Proposition 2.20. *There is a canonical homomorphism*

$$\Psi : T(\mathbb{Q}_p)/T_l(\mathbb{Q}_p) \to \mathrm{H}^1_{\mathrm{dR}}(X)$$

with the following properties:

(1) Ψ is the identity on differentials of the first kind;
(2) Ψ sends third kind differentials to second kind differentials modulo exact differentials.

Proof. See [CG89, §2]. □

Definition 2.21. Let $D \in \mathrm{Div}^0(X)$. Then we define ω_D to be the unique differential of the third kind with $\mathrm{Res}(\omega_D) = D$ and $\Psi(\omega_D) \in W$.

In fact, Ψ can be extended to general meromorphic (and even rigid analytic) forms. Having fixed our normalized differential ω_D, we can now define:

Definition 2.22. The *local height at p* of $D_1, D_2 \in \mathrm{Div}^0(X)$ with disjoint support is

$$h_p(D_1, D_2) = \int_{D_2} \omega_{D_1}.$$

As in §2.1, we can take W to be the unit root subspace if p is ordinary. It can be approximated to any desired p-adic precision as follows:

Proposition 2.23. *If ϕ is a lift of Frobenius, then $\{(\phi^*)^n\omega_g, \cdots, (\phi^*)^n\omega_{2g-1}\}$ is a basis for the unit root subspace modulo p^n.*

Proof. See the proof of [BB12, Proposition 6.1], which is stated for odd degree hyperelliptic curves, but which carries through in the context of any nice curve. □

Remark 2.24. For applications to quadratic Chabauty, there is no need to use the unit root subspace; any isotropic subspace will do. In particular, one does not need to restrict to ordinary primes. For other applications, however, the height with respect unit root subspace is preferable, since it is equivalent to both the canonical Mazur-Tate height by [Col91] and the Schneider height by [MT83, Proposition 4.4]. The corresponding p-adic regulator appears in a generalization of Conjecture 2.6, the p-adic BSD

conjecture of Mazur–Tate–Teitelbaum, due to the authors and Stein [BMS16]. When the reduction is not good ordinary, the relation between the different constructions of p-adic heights need not be equivalent, see for instance [Wer98].

For the following algorithm, recall that $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_p)$ is equipped with the cup product: a canonical, alternating, non-degenerate bilinear form, which we compute using Serre's formula:

$$\begin{aligned}\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_p) \times \mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_p) &\to \mathbb{Q}_p \\ ([\mu_1],[\mu_2]) &\mapsto [\mu_1 \cup \mu_2] = \sum_{Q \in X(\mathbb{C}_p)} \mathrm{Res}_Q\left(\mu_2 \int \mu_1\right).\end{aligned}$$

Remark 2.25. Note that since μ_2 is of the second kind, it has residue zero everywhere, and so the result above does not depend on a choice of constant of integration for $\int \mu_1$.

Algorithm 2.26 (Coleman integral of differentials of the third kind [BB12])**.**
Input:

- A differential ω with $\mathrm{Res}(\omega) = (P) - (Q)$ such that $P, Q \in X(\mathbb{Q}_p)$ are non-Weierstrass points.
- Points $R, S \in X(\mathbb{Q}_p)$ such that R, S do not lie in residue disk of P, Q.

Output: The integral $\int_S^R \omega$.

(1) Compute $\Psi(\omega) \in \mathrm{H}^1_{\mathrm{dR}}(X)$ by determining the coefficients b_i in $\Psi(\omega) = \sum_{i=0}^{2g-1} b_i[\omega_i]$. For instance, this can be done by computing the cup products $\Psi(\omega) \cup [\omega_j]$ for all j and setting up a linear system.
(2) Let $\alpha := \phi^*\omega - p\omega$. Use Frobenius equivariance to compute
$$\Psi(\alpha) = \phi^*\Psi(\omega) - p\Psi(\omega).$$
Note that $\phi^*\Psi(\omega)$ can be computed using the matrix of Frobenius with respect to the chosen basis of $\mathrm{H}^1_{\mathrm{dR}}(X)$.
(3) Let β be a differential 1-form with $\mathrm{Res}(\beta) = (R) - (S)$. As in Step (1) above, compute $\Psi(\beta)$.
(4) Compute $\Psi(\alpha) \cup \Psi(\beta)$. (This is easy, since both are elements in $\mathrm{H}^1_{\mathrm{dR}}(X)$ that we have computed.)
(5) Compute $\int_{\phi(S)}^{S} \omega$ and $\int_R^{\phi(R)} \omega$. (These are tiny integrals.)
(6) Compute $\sum_{A \in X(\overline{\mathbb{Q}_p})} \mathrm{Res}_A\left(\alpha \int \beta\right)$. (This may be more involved since there might be poles that are not defined over $\mathbb{Q}_p$.)
(7) Finally, we get
$$\int_S^R \omega = \frac{1}{1-p}\left(\Psi(\alpha) \cup \Psi(\beta) + \sum_{A \in X(\overline{\mathbb{Q}_p})} \mathrm{Res}_A\left(\alpha \int \beta\right) - \int_{\phi(S)}^{S} \omega - \int_R^{\phi(R)} \omega\right).$$

Remark 2.27. We introduce the auxiliary differential α in Step (1) above, because α is almost of the second kind, meaning that the sum of residues of α in each annulus in 0.

Algorithm 2.28 (The local height at p of the global p-adic height, $h_p(D_1, D_2)$ [BB12])**.**

(1) Let ω be a differential in $T(\mathbb{Q}_p)$ with $\mathrm{Res}(\omega) = D_1$.
(2) Compute $\Psi(\omega) = \sum_{i=0}^{2g-1} a_i[\omega_i] \in \mathrm{H}^1_{\mathrm{dR}}(X)$. Then $\Psi(\omega) - \sum_{i=0}^{g-1} a_i[\omega_i] \in W$. Let
$$\omega_{D_1} = \omega - \sum_{i=0}^{g-1} a_i\omega_i.$$

(3) Compute using Algorithm 2.26

$$h_p(D_1, D_2) = \int_{D_2} \omega_{D_1}.$$

Remark 2.29. For a discussion of the precision needed in Algorithm 2.28, see [BB12, §6.2].

Example 2.30 ([BBM17, Example 9.2]). Consider the genus 3 curve

$$X : y^2 = (x^3 + x + 1)(x^4 + 2x^3 - 3x^2 + 4x + 4).$$

This is a new modular curve C_{496}^J studied by Baker–González-Jiménez–González–Poonen [BGJGP05]. Let

$$P = (-1, 2), Q = (0, 2), R = (-2, 12), S = (3, 62) \in X(\mathbb{Q})$$

and let w denote the hyperelliptic involution.

We take $p = 17$ and use Algorithm 2.28 to compute the local height $h_{17}(D_2, D_3)$ where $D_2 = (S) - (w(Q))$ and $D_3 = (w(S)) - (R)$. Let ω be the differential (19) constructed in Example 2.19 using residue divisor D_2. Using Algorithm 2.26, we find

$$\int_{D_3} \omega = 14 \cdot 17 + 12 \cdot 17^2 + 12 \cdot 17^3 + 4 \cdot 17^4 + 6 \cdot 17^5 + 13 \cdot 17^6 + 11 \cdot 17^7 + 15 \cdot 17^8 + 6 \cdot 17^9 + O(17^{10}).$$

Let $\eta := \sum_{i=0}^{2} a_i \omega_i$ where $\Psi(\omega) = \sum_{i=0}^{5} a_i [\omega_i]$. We calculate that

$$\begin{aligned}\eta = {} & (11 + 8 \cdot 17 + 2 \cdot 17^3 + 11 \cdot 17^4 + 9 \cdot 17^6 + 14 \cdot 17^7 + 14 \cdot 17^8 + 2 \cdot 17^9 + O(17^{10}))\omega_0 + \\ & (15 + 2 \cdot 17 + 17^2 + 9 \cdot 17^3 + 15 \cdot 17^4 + 2 \cdot 17^5 + 2 \cdot 17^6 + 14 \cdot 17^8 + 10 \cdot 17^9 + O(17^{10}))\omega_1 + \\ & (12 + 4 \cdot 17^2 + 12 \cdot 17^3 + 13 \cdot 17^4 + 12 \cdot 17^5 + 15 \cdot 17^6 + 14 \cdot 17^7 + 10 \cdot 17^8 + O(17^{10}))\omega_2,\end{aligned}$$

and using Algorithm 1.37, we find

$$\begin{aligned}\int_{D_3} \omega_0 &= 17 + 5 \cdot 17^3 + 9 \cdot 17^4 + 17^5 + 12 \cdot 17^6 + 16 \cdot 17^7 + 8 \cdot 17^8 + 3 \cdot 17^9 + O(17^{10}) \\ \int_{D_3} \omega_1 &= 14 \cdot 17 + 14 \cdot 17^2 + 4 \cdot 17^3 + 5 \cdot 17^4 + 16 \cdot 17^5 + 11 \cdot 17^6 + 4 \cdot 17^9 + O(17^{10}) \\ \int_{D_3} \omega_2 &= 7 \cdot 17 + 4 \cdot 17^2 + 4 \cdot 17^3 + 7 \cdot 17^5 + 7 \cdot 17^6 + 12 \cdot 17^7 + 9 \cdot 17^8 + O(17^{10}),\end{aligned}$$

so we have

$$\int_{D_3} \eta = 16 \cdot 17 + 5 \cdot 17^2 + 13 \cdot 17^3 + 2 \cdot 17^4 + 16 \cdot 17^5 + 14 \cdot 17^6 + 16 \cdot 17^7 + 9 \cdot 17^8 + 17^9 + O(17^{10})).$$

Putting this together, we have

$$h_{17}(D_2, D_3) = \int_{D_3} \omega - \int_{D_3} \eta = 15{\cdot}17 + 6{\cdot}17^2 + 16{\cdot}17^3 + 17^4 + 7{\cdot}17^5 + 15{\cdot}17^6 + 11{\cdot}17^7 + 5{\cdot}17^8 + 5{\cdot}17^9 + O(17^{10}).$$

Likewise we may compute $h_{17}(D_3, D_2) = \int_{D_2} \omega_{D_3}$ and numerically verify that $h_{17}(D_2, D_3) = h_{17}(D_3, D_2)$.

Using `SageMath` code available on GitHub [Bal], here is how to compute the final local height values at 17:

```
R.<x> = QQ[]
X = HyperellipticCurve((x^3+x+1)*(x^4 +2*x^3-3*x^2+4*x+4))
K = Qp(17,10)
XK = X.change_ring(K)
```

```
S = XK(3,62)
iS = XK(3,-62)
iQ = XK(0,-2)
R = XK(-2,12)
XK.height([(1,S),(-1,iQ)],[(1,iS),(-1,R)])
//15*17 + 6*17^2 + 16*17^3 + 17^4 + 7*17^5 + 15*17^6 + 11*17^7 + 5*17^8 + 5*17^9 + O(17^10)
XK.height([(1,iS),(-1,R)],[(1,S),(-1,iQ)])
//15*17 + 6*17^2 + 16*17^3 + 17^4 + 7*17^5 + 15*17^6 + 11*17^7 + 5*17^8 + 5*17^9 + O(17^10)
```

One can also use `Magma` code for this, see [BDM+a].

Remark 2.31. Forthcoming work of Gajović and Müller will give an alternative algorithm for the local Coleman–Gross height on hyperelliptic curves, without any assumption on the existence of a $\mathbb{Q}_p$-rational Weierstrass point. This algorithm has more restrictions on the divisors D_1, D_2, but is usually faster when it applies. An extension to superelliptic curves is work in progress.

Remark 2.32. The construction of Coleman–Gross local heights can be extended to curves with bad reduction, replacing Coleman integration with Vologodsky integration; see for instance [Bes22]. Forthcoming work of Bianchi, Kaya, and Müller will discuss an extension of Algorithm 2.28 to hyperelliptic curves with semistable reduction and a comparison with the local heights constructed by Bianchi (see Remark 2.14) when the genus is 2, extending [BB15, Theorem 4.1], which is for elliptic curves.

Since the global p-adic height respects linear equivalence, it can be extended to pairs of divisors with common support. The local height pairings can also be extended in a non-canonical way. We first discuss this for the pairing at p. This uses the following idea of Gross [Gro86]. Let t be a section of the tangent bundle of X. In particular, this gives, at each point x in the common support of our divisors, a basis t_x of the tangent space. Let $z := z_x$ be a uniformizing parameter at x with $\partial_{t_x} z = 1$. Any rational function f on $X_{\mathbb{Q}_p}$ then has a well-defined value at x,

$$f[x] = \frac{f}{z^m}(x)$$

where m is the order of f at x. This depends only on t, but not on z.

For an odd degree hyperelliptic curve, we let ω_i be $\frac{x^i dx}{2y}$, and we let $\{\overline{\omega}_i\}$ be the dual basis to $\{\omega_i\}_{i=0,\dots,g-1}$ with respect to the cup product pairing. We define a section of the tangent bundle by the dual of ω_0 away from ∞ and by ω_{g-1} at ∞.

Proposition 2.33 ([BB15, BBM16])**.** *Let $X/\mathbb{Q}$ be a hyperelliptic curve given by a monic odd degree model. With the choice of t above, the local height $h_p((P)-(\infty),(P)-(\infty))$ can be written as a double integral*

$$h_p((P)-(\infty),(P)-(\infty)) = -2\sum_{i=0}^{g-1}\int_b^P \omega_i\overline{\omega}_i,$$

where b is a tangential base point at ∞ (see §1.6).

The proof uses p-adic Arakelov theory, as developed by Besser [Bes05]. In this theory, the local height is given by a p-adic Green function, similar to classical Arakelov theory. One shows equality in the proposition by first computing the curvature of this p-adic Green function, then proving that the two values are the same up to a constant, which one can then show to be 0.

As a consequence, we have that

$$\theta(z) := h_p((z)-(\infty),(z)-(\infty))$$

extends to a locally analytic function on $X(\overline{\mathbb{Q}}_p)$ away from the disk at infinity, where we fix the choice of tangent vectors as in Proposition 2.33.

2.3. **An application to integral points.** For certain curves, we can use p-adic heights to study integral points.

Theorem 2.34 (Quadratic Chabauty for integral points on hyperelliptic curves [BBM16, Theorem 3.1])**.** *Let $f(x) \in \mathbb{Z}[x]$ be a monic separable polynomial of degree $2g+1 \geq 3$. Let $\mathscr{U} = \operatorname{Spec}(\mathbb{Z}[x,y]/(y^2 - f(x))$ and let X be the normalization of the projective closure of the generic fiber of $\mathscr{U}$. Let J be the Jacobian of X and assume that $\operatorname{rk} J(\mathbb{Q}) = g$. Choose a prime p of good reduction and suppose that $\log\colon J(\mathbb{Q}) \otimes \mathbb{Q}_p \to \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ is an isomorphism*[11]*. Then there exist explicitly computable constants $\alpha_{ij} \in \mathbb{Q}_p$ such that the function*

$$\rho(z) = \theta(z) - \sum_{0 \leq i \leq j \leq g-1} \alpha_{ij} \int_\infty^z \omega_i \int_\infty^z \omega_j$$

takes values in an explicitly computable finite set $\mathcal{S} \subset \mathbb{Q}_p$ for all z in $\mathscr{U}(\mathbb{Z}[\frac{1}{p}])$.

Proof. The key idea is that the global height $h((P)-(\infty),(P)-(\infty))$ can be decomposed in two ways:

(i) Because of the assumption on log, and since the global height h is a symmetric bilinear pairing we can find $\alpha_{ij} \in \mathbb{Q}_p$ such that for all $P \in X(\mathbb{Q})$ we have

$$h((P)-(\infty),(P)-(\infty)) = \sum \alpha_{ij} \int_\infty^P \omega_i \int_\infty^P \omega_j,$$

and this extends to a locally analytic function on $X(\bar{\mathbb{Q}}_p)$ outside the residue disk of ∞.

(ii) We have

$$h((P)-(\infty),(P)-(\infty)) = \theta(P) + \sum_{\ell \neq p} h_\ell((P)-(\infty),(P)-(\infty)).$$

Hence we deduce

$$\rho(P) = -\sum_{\ell \neq p} h_\ell((P)-(\infty),(P)-(\infty))$$

for all $P \in X(\mathbb{Q})$. The proof of the theorem now follows from

Proposition 2.35 ([BBM16, Proposition 3.3])**.** *Let $\ell \neq p$ be prime. There is a proper regular model $\mathcal{X}$ of $X \otimes \mathbb{Q}_\ell$ over $\mathbb{Z}_\ell$ such that if $z \in X(\mathbb{Q}_\ell)$ is integral then $h_\ell((z)-(\infty),(z)-(\infty))$ depends solely on the component of the special fiber $\mathcal{X}_\ell$ that the section in $\mathcal{X}(\mathbb{Z}_\ell)$ corresponding to z intersects, and is explicitly computable. If the sections corresponding to z and ∞ intersect the same component, then the local height is 0.*

We will not prove this here, but see §2.3.1 below. □

As a special case of Theorem 2.34, we have the following extension of Theorem 1.68.

Corollary 2.36. *Let f, p, $\mathscr{U}$ and X satisfy the conditions of Theorem 2.34. Suppose that there exists a proper regular model $\mathcal{X}$ / $\mathbb{Z}$ of X such that, for every bad prime ℓ, all $\mathbb{Q}_\ell$-rational points on X reduce to the same irreducible component of the special fiber of $\mathcal{X} \otimes \mathbb{Z}_\ell$. Then there exist explicitly computable constants $\alpha_{ij} \in \mathbb{Q}_p$ such that the function*

$$\rho(z) = \theta(z) - \sum_{0 \leq i \leq j \leq g-1} \alpha_{ij} \int_\infty^z \omega_i \int_\infty^z \omega_j$$

[11] If this fails, we can simply use Chabauty–Coleman to compute the rational points.

vanishes on $\mathscr{U}(\mathbb{Z}[\frac{1}{p}])$.

We can use Theorem 2.34 to compute the integral points on a curve X satisfying the conditions in practice. We give some more computational details here; for more, see [BBM17]. For an extension to number fields, see [BBBM21]. For $P \in J(\mathbb{Q}_p)$ and $i \in \{0, \dots, g-1\}$, we set $f_i(P) := \int_0^P \omega_i$; then $f_0, \dots, f_{g-1}$ restrict to linearly independent functionals on $J(\mathbb{Q}) \otimes \mathbb{Q}_p$ by assumption.

Algorithm 2.37 (The set of integral points on a curve $X/\mathbb{Q}$ satisfying the assumptions of Theorem 2.34)**.**

(1) Let $D_1, \dots, D_g \in \mathrm{Div}^0(X)$ be representatives of a basis for $J(\mathbb{Q}) \otimes \mathbb{Q}$. Then compute the global height pairings $h(D_i, D_j)$. A basis for the space of bilinear forms on $J(\mathbb{Q}) \otimes \mathbb{Q}$ is given by $1/2(f_k f_\ell + f_\ell f_k)$ so compute $1/2(f_k(D_i)f_\ell(D_j) + f_\ell(D_i)f_k(D_j))$ and do linear algebra to compute $\alpha_{k\ell}$:

$$h(D_i, D_j) = \sum_{0 \le k, \ell \le g-1} \alpha_{k\ell}(1/2(f_k(D_i)f_\ell(D_j) + f_\ell(D_i)f_k(D_j))).$$

(2) In order to compute $\{\overline{\omega}_i\}$ for $0 \le i \le g-1$ such that $[\overline{\omega}_i] \cup [\omega_j] = \delta_{ij}$ we proceed as follows:
   (i) Compute a splitting of $\mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p}) = \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1) \oplus W$, isotropic with respect to the cup product. For instance, when p is ordinary we can take W to be the unit root eigenspace of Frobenius. In this case, modulo p^n, a basis for W is given by $\{(\phi^*)^n\omega_g, \dots, (\phi^*)^n\omega_{2g-1}\}$.
   (ii) For $j = 0, \dots, g-1$, let $\widetilde{\omega_j}$ be a projection on W along $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)$, i.e., $\widetilde{\omega_j} = \omega_j - \sum_{i=0}^{g-1} a_i\omega_i$ for some $a_i \in \mathbb{Q}_p$.
   (iii) Use the cup product matrix to compute

$$\overline{\omega}_j = \sum_{i=g}^{2g-1} b_{ji}\widetilde{\omega_i}$$

   for $j = 0$ to $g-1$.

(3) Expand $\theta(z) := -2\sum_{i=0}^{g-1} \int \omega_i\overline{\omega}_i$ into a power series in each residue disk D not containing ∞, compute a $\mathbb{Z}_p$-point $P \in D$, the value $\theta(P)$, and a local coordinate z_P at P. Then

$$\theta(z) = -2\sum_{i=0}^{g-1}\int_b^{g-1} \omega_i\overline{\omega}_i = -2\left(\sum_{i=0}^{g-1}\int_b^P \omega_i\overline{\omega}_i + \sum_{i=0}^{g-1}\int_P^{z_P} \omega_i\overline{\omega}_i + \sum_{i=0}^{g-1}\int_P^{z_P} \omega_i \int_b^P \overline{\omega}_i\right)$$

   which is equal to

$$\theta(P) - 2\left(\sum_{i=0}^{g-1}\int_P^{z_P} \omega_i\overline{\omega}_i + \sum_{i=0}^{g-1}\int_P^{z_P} \omega_i \int_b^P \overline{\omega}_i\right),$$

   where b is a tangential basepoint at infinity. Note that, aside from computing the value of $\theta(P)$, this only uses tiny double integrals in the disk of P.

(4) Use intersection theory to compute the finite set $\mathcal{S}_\ell$ of possible values of $-h_\ell((P) - (\infty), (P) - (\infty))$ for bad primes ℓ and integral $P \in X(\mathbb{Q}_\ell)$. Obtain a finite set $\mathcal{S} \subset \mathbb{Q}_p$ such that $\sum_{\ell \ne p} -h_\ell((P) - (\infty), (P) - (\infty)) \in \mathcal{S}$ for $P \in \mathscr{U}(\mathbb{Z}[\frac{1}{p}])$.

(5) Now proceed similar to the classical Chabauty–Coleman method: We can expand ρ in each disk, set it equal to each value in $\mathcal{S}$, solve for all $z \in \mathscr{U}(\mathbb{Z}_p)$ such that $\rho(z) \in \mathcal{S}$. Take the collection of all such points, we will call that solution set $\mathcal{Z}$. If we find solutions of multiplicity greater than 1, throw an error.

(6) If $\mathcal{Z}$ is strictly larger than the known points in $\mathscr{U}(\mathbb{Z})$, run the Mordell–Weil sieve [BS10] (see also §2.3.2), possibly after re-running steps (1)–(5) on a collection of good primes p. If this fails to show that the known points are exactly $\mathscr{U}(\mathbb{Z})$, and does not recover any new elements of $\mathscr{U}(\mathbb{Z})$, throw an error.

See [BBM17, Algorithm 8.1] for details, improvements and strategies for dealing with the cases when this algorithm throws an error.

2.3.1. *Local heights away from p.* We say a few more words about the local heights at $\ell \neq p$. Given divisors $D_1, D_2 \in \mathrm{Div}^0(X)$ with disjoint support, we can express $h_\ell(D_1, D_2)$ as an intersection multiplicity

$$h_\ell(D_1, D_2) = (\mathcal{D}_1 \cdot \mathcal{D}_2)\chi_\ell(\ell)\,, \tag{20}$$

where $\chi_\ell(\ell) = -\log_p(\ell)$, see the beginning of the present subsection. Here $\mathcal{D}_i$ is an extension of D_i to a regular model $\mathcal{X}$ of $X_{\mathbb{Q}_\ell}$ such that $\mathcal{D}_i$ has trivial intersection with all vertical divisors. See [CG89, Proposition 1.2]. The argument that (20) satisfies the properties of Definition 2.18 and that it is unique are exactly as in the case of real-valued heights, see [Lan88, Chapter III].

In practice, we use a strong desingularization $\mathcal{X}$ of the Zariski closure of X in $\mathbb{P}^2_{\mathbb{Z}_\ell}(1, g+1, 1)$. We can extend the local height pairing to divisors with common support using the same approach that we used for the local heights at p, see the discussion before Proposition 2.33. If we pick the same section of the tangent bundle as the one chosen there, then we find that for $P \in X(\mathbb{Q}_\ell)$,

$$h_\ell((P) - (\infty), (P) - (\infty)) = \left(2(P_{\mathcal{X}} \cdot \infty_{\mathcal{X}}) + (P_{\mathcal{X}} \cdot V) - \Phi((P) - (\infty))^2\right)\chi_\ell(\ell)\,, \tag{21}$$

where

- $P_{\mathcal{X}} \in \mathcal{X}(\mathbb{Z}_\ell)$ is the section corresponding to P, and likewise for $\infty_{\mathcal{X}}$,
- V is the vertical part of $\mathrm{div}\left(\frac{dx}{2y}\right) \in \mathrm{Div}(\mathcal{X})$,
- $\Phi((P) - (\infty)) \in \mathrm{Div}(\mathcal{X}) \otimes \mathbb{Q}$ is vertical such that $P_{\mathcal{X}} - \infty_{\mathcal{X}} + \Phi((P) - (\infty))$ has intersection multiplicity 0 with all vertical divisors on $\mathcal{X}$.

See Lemma 3.4 and the proof of Proposition 3.3 of [BBM16]. The latter two terms on the right hand side of (21) only depends on the irreducible component of the special fiber $\mathcal{X}_\ell$ that $P_{\mathcal{X}}$ intersects. The first term $2(P_{\mathcal{X}} \cdot \infty_{\mathcal{X}})$ vanishes if P is integral. In general, it is determined by the denominator of $x(P)$, which we are unable to control (otherwise we would have a height bound and effective version of Faltings's theorem). This makes it difficult to go beyond integral points using this construction. We will see later in Section 5 that when J has Picard number > 1, then we can control the local heights at $\ell \neq p$ for a different pair of divisors depending on $\mathbb{Q}_\ell$-*rational points*. This will allow us to compute $X(\mathbb{Q})$ in favorable circumstances.

Example 2.38. In the case of elliptic curves, we have the Mazur–Stein–Tate p-adic height

$$h(P) = \frac{1}{p}\log_p(\sigma(P)) - \frac{1}{p}\log_p(d(P))$$

on a certain finite index subgroup of the Mordell–Weil group, as well as the Coleman–Gross p-adic height

$$h((P) - (\infty)) = h_p((P) - (\infty)) + \sum_{v \neq p} h_v((P) - (\infty)).$$

Extending appropriately, we have $\frac{1}{p}\log(\sigma(P)) = h_p((P) - (\infty))$ and $-\frac{1}{p}\log_p(d(P)) = \sum_{\ell \neq p} h_\ell((P) - (\infty))$.

To compute local heights h_ℓ for $\ell \neq p$, we need to compute regular models (implemented in `Magma` by Donnelly, see also recent work of Dokchitser [Dok21]) and Gröbner bases of ideals of divisors. For more details, see [Hol12, Mül14, VBHM20].

Example 2.39. Let $X\colon y^2 = (x^3+x+1)(x^4+2x^3-3x^2+4x+4)$ be the new modular curve C_{496}^J discussed in Example 2.30. The discriminant of X factors as $2^{24}\cdot 31^4$, so the bad primes are 2 and 31. Hence the elements of the set $\mathcal{S}$ in Theorem 2.34 are of the form $\alpha+\beta$, where $\alpha \in \mathcal{S}_2 = \{h_2((P)-(\infty),(P)-(\infty)) : P \in \mathscr{U}(\mathbb{Z}_2)\}$ and $\beta \in \mathcal{S}_{31} = \{h_{31}((P)-(\infty),(P)-(\infty)) : P \in \mathscr{U}(\mathbb{Z}_{31})\}$. We briefly discuss how to compute the set $\mathcal{S}_{31}$. The model $\mathcal{X}'$ defined in $\mathbb{P}^2_{\mathbb{Z}_{31}}(1,3,1)$ by the given equation of X is regular outside the point $(3,0)$ on its special fiber $\mathcal{X}'_{31}$ (the points $(14,0)$ and $(23,0)$ are also singular points of the special fiber, but they are regular on $\mathcal{X}'$). By blowing up $\mathcal{X}'$ in $(3,0) \in \mathcal{X}_{31}$, we obtain a regular model $\mathcal{X}$ whose special fiber consists of two rational curves Γ_0 and Γ_1 of multiplicity 1, intersecting transversally in two points. This model is semistable, but we don't need that. The component Γ_0 has two nodes and contains the reductions of all regular points on $X(\mathbb{Q}_{31})$. To compute $\mathcal{S}_{31}$, we need the intersection matrix

$$M_{31} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}. \tag{22}$$

This information can be obtained using `Magma` as follows:

```
C := HyperellipticCurve((x^3+x+1)*(x^4+2*x^3-3*x^2+4*x+4));
C31 := RegularModel(C, 31);
M31 := IntersectionMatrix(C31);
Multiplicities(M31); \\ 1, 1
```

The `RegularModel` package in `Magma` due to Steve Donnelly always computes a strong desingularization. It requires a smooth model in a (weighted) projective space. Alternatively, one can use the `Sage` package `cluster-pictures` due to Alex Best and Raymond van Bommel, available from https://alexjbest.github.io/cluster-pictures. This implements algorithms for the local arithmetic of hyperelliptic curves in residue characteristic $\neq 2$ from [DDMM22], see also [BBB+22].

The divisor of the differential $\frac{dx}{2y}$ on $\mathcal{X}$ is $2\infty_{\mathcal{X}}$, so that we only need to compute the final term in (21). Let $P \in \mathscr{U}(\mathbb{Z}_\ell)$. If P reduces to Γ_0 (which also intersects $\infty_{\mathcal{X}}$)), then $P_{\mathcal{X}} - \infty_{\mathcal{X}}$ intersects both Γ_0 and Γ_1 trivially; hence we can take $\Phi((P)-(\infty)) = 0$. If P reduces to Γ_1, then we can take $\Phi((P)-(\infty)) = \frac{1}{2}\Gamma_1$, since then $(\Phi((P)-(\infty))\cdot\Gamma_0) = 1 = -(P_{\mathcal{X}}-\infty_{\mathcal{X}}\cdot\Gamma_0)$ and $(\Phi((P)-(\infty))\cdot\Gamma_1) = -1$, as desired. It follows that $\mathcal{S}_{31} = \{0, \frac{1}{2}\log_p 31\}$.

The set $\mathcal{S}_2$ is more difficult to compute, since the regular model one obtains by blowing up is more complicated; its special fiber has 13 irreducible components and the intersection matrix is

$$M_2 = \begin{pmatrix} -6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

where the first 6 components have multiplicity 1 and the final 7 have multiplicity 2. For this computation, we can use `Magma`, but not `cluster-pictures`, since the latter only works in residue characteristic $\neq 2$. We leave it to the reader to show that

$$\mathcal{S}_2 = \left\{0, \log_p 2, \frac{5}{4}\log_p 2, \frac{7}{4}\log_p 2\right\}.$$

2.3.2. *The Mordell–Weil sieve.* We briefly review the Mordell–Weil sieve. The original idea is due to Scharaschkin [Sch99], see [BS10] for an implementation-oriented account. Let $M > 1$ be an integer, let U be a finite set of primes of good reduction for X and consider the commutative diagram

$$\begin{array}{ccc} X(\mathbb{Q}) & \longrightarrow & J(\mathbb{Q})/MJ(\mathbb{Q}) \\ \downarrow & & \downarrow{\scriptstyle \alpha_U} \\ \prod_{v\in U} X(\mathbb{F}_v) & \xrightarrow[\beta_U]{} & \prod_{v\in U} J(\mathbb{F}_v)/MJ(\mathbb{F}_v)\,, \end{array}$$

where $\beta_U = \prod_{v\in U}\beta_v$ (coming from a chosen embedding $X \to J$) and $\alpha = \prod_{v\in U}\alpha_v$. Commutativity gives us a way to exclude the existence of rational points satisfying certain local conditions, by choosing the set U and the integer M carefully. When $X(\mathbb{Q})$ is empty, heuristics due to Poonen [Poo06] predict that there should always be a choice of U and M so that the images of α_U and β_U do not intersect.

We can use the Mordell–Weil sieve to show that for a fixed prime p, a given residue class in $X(\mathbb{Q}_p)$ does not contain a rational point. For this application we choose a modulus $M = M' \cdot p$ for some auxiliary integer M' and we choose U to be a set containing the prime divisors q of pM', and additional primes ℓ of good reduction with the property that $\gcd(\#J(\mathbb{F}_\ell), \#J(\mathbb{F}_q))$ is large for some of these q. This approach can be combined with Chabauty-Coleman as in [PSS07]. For a different approach to combining Chabauty-Coleman with the Mordell–Weil sieve, see [BS10, §4.4]; the idea described there is implemented in `Magma` for genus 2 curves with Jacobian rank 1 and can be called using the command `Chabauty`.

In the quadratic Chabauty setting, where we assume that $r = g$, we can also apply the sieve in a different way. The discussion here is relevant both for the quadratic Chabauty method for integral points discussed in the present section, as well as the quadratic Chabauty method for rational points presented in Section 5 below. Suppose that we have a point $P \in X(\mathbb{Q}_p)$, computed to finite precision p^N, and we want to show that it does not come from a rational point. Assume that it does, so there

are integers $a_1, \dots, a_g$ such that

$$[(P) - (b)] = a_1 P_1 + \dots + a_g P_g,$$

where $P_1, \dots, P_g$ generate $J(\mathbb{Q}) \otimes \mathbb{Q}$ and $b \in X(\mathbb{Q})$ is a base point for the Abel-Jacobi map. Assume, for simplicity, that $J(\mathbb{Q})_{\text{tors}}$ is trivial. We can compute, for instance using linearity of single Coleman integrals, a tuple $(\tilde{a}_1, \dots, \tilde{a}_g) \in (\mathbb{Z}/p^N\mathbb{Z})^g$ such that $a_i \equiv \tilde{a}_i \pmod{p^N}$ for all $i \in \{1, \dots, g\}$. We can then use the Mordell–Weil sieve to show that the corresponding coset of $p^N J(\mathbb{Q})$ does not contain the image of a rational point on X. We can also apply quadratic Chabauty with several primes p or choose $M = p^n M'$, where M' is a small auxiliary integers as above, as in Example 2.40 below. For more details on the combination of quadratic Chabauty and the Mordell–Weil sieve see [BBM17], [BBB+21, §6.7].

2.3.3. *A worked example.*

Example 2.40 ([BBM17, Example 9.2]). Let $X\colon y^2 = (x^3 + x + 1)(x^4 + 2x^3 - 3x^2 + 4x + 4)$ be the new modular curve C_{496}^J discussed in Example 2.30 and Example 2.39. We want to use Algorithm 2.37 to show that up to the hyperelliptic involution w, the points

$$P = (-1, 2)\,,\ Q = (0, 2)\,,\ R = (-2, 12)\,,\ S = (3, 62) \in X(\mathbb{Q})$$

are the only integral points. We go through Steps (1)–(6) of Algorithm 2.37 for $p = 17$.

Step (1): For this curve, we can use `Magma's RankBounds` to compute $\operatorname{rk} J(\mathbb{Q}) = 3$, via 2-descent as described by Stoll in [Sto01]. Generators for $J(\mathbb{Q}) \otimes \mathbb{Q}$ are given by

$$P_1 = [(P) - (\infty)]\,,\ P_2 = [(S) - (w(Q))]\,,\ P_3 = [(w(S)) - (R)]\,.$$

This can be shown by computing heights or by reducing modulo sufficiently many primes.

We discussed the computation of the 17-adic local heights $h_{17}(D_2, D_3)$ in Example 2.30, where $D_2 = (S) - (w(Q))$ and $D_3 = (w(S)) - (R)$. We may compute $h_\ell(D_2, D_3)$ for all primes $\ell \neq 17$ using (20) via an algorithm of van Bommel, Holmes and Müller [VBHM20]. After moving divisors, we can, alternatively, use the `Magma` command `LocalIntersectionData`, which implements the techniques of [Mül14]. For the latter, we need divisors of the form $\sum_i (P_i) - n(\infty)$ or $\sum_i (P_i) - \sum_j ((Q_j) + (w(Q_j)))$.

Step (2): Since $p = 17$ is ordinary, we take W to be the unit root subspace. Modulo 17^{10}, a basis for W is given by We compute $\{\overline{\omega_0}, \overline{\omega_1}, \overline{\omega_2}\}$, the dual basis to $\{\omega_0, \omega_1, \omega_2\}$ with respect to the cup product pairing. These are given as vectors with respect to the basis $\{\omega_0, \omega_1, \dots, \omega_5\}$.

$$\begin{aligned}
\overline{\omega_0} = (&14 + 5\cdot 17 + 3\cdot 17^2 + 15\cdot 17^3 + 17^4 + 14\cdot 17^5 + 15\cdot 17^6 + 13\cdot 17^7 + 11\cdot 17^8 + O(17^{10}),\\
&14 + 8\cdot 17 + 17^2 + 2\cdot 17^4 + 4\cdot 17^5 + 2\cdot 17^6 + 13\cdot 17^7 + 2\cdot 17^9 + O(17^{10}),\\
&5\cdot 17 + 6\cdot 17^2 + 17^3 + 6\cdot 17^4 + 6\cdot 17^5 + 13\cdot 17^7 + 8\cdot 17^8 + 11\cdot 17^9 + O(17^{10}),\\
&6 + O(17^{10}),\\
&9 + 16\cdot 17 + 16\cdot 17^2 + 16\cdot 17^3 + 16\cdot 17^4 + 16\cdot 17^5 + 16\cdot 17^6 + 16\cdot 17^7 + 16\cdot 17^8 + 16\cdot 17^9 + O(17^{10}),\\
&12 + 16\cdot 17 + 16\cdot 17^2 + 16\cdot 17^3 + 16\cdot 17^4 + 16\cdot 17^5 + 16\cdot 17^6 + 16\cdot 17^7 + 16\cdot 17^8 + 16\cdot 17^9 + O(17^{10}))
\end{aligned}$$

$$\begin{aligned}\overline{\omega_1} = (&9\cdot 17 + 17^2 + 2\cdot 17^4 + 4\cdot 17^5 + 2\cdot 17^6 + 13\cdot 17^7 + 2\cdot 17^9 + O(17^{10}),\\ &5 + 10\cdot 17^2 + 6\cdot 17^3 + 5\cdot 17^4 + 8\cdot 17^5 + 2\cdot 17^6 + 8\cdot 17^8 + 14\cdot 17^9 + O(17^{10}),\\ &6 + 10\cdot 17^2 + 14\cdot 17^3 + 8\cdot 17^4 + 11\cdot 17^5 + 12\cdot 17^6 + 16\cdot 17^7 + 12\cdot 17^8 + 2\cdot 17^9 + O(17^{10}),\\ &13 + 16\cdot 17 + 16\cdot 17^2 + 16\cdot 17^3 + 16\cdot 17^4 + 16\cdot 17^5 + 16\cdot 17^6 + 16\cdot 17^7 + 16\cdot 17^8 + 16\cdot 17^9 + O(17^{10}),\\ &14 + 16\cdot 17 + 16\cdot 17^2 + 16\cdot 17^3 + 16\cdot 17^4 + 16\cdot 17^5 + 16\cdot 17^6 + 16\cdot 17^7 + 16\cdot 17^8 + 16\cdot 17^9 + O(17^{10}),\\ &0)\end{aligned}$$

$$\begin{aligned}\overline{\omega_2} = (&14 + 5\cdot 17 + 6\cdot 17^2 + 17^3 + 6\cdot 17^4 + 6\cdot 17^5 + 13\cdot 17^7 + 8\cdot 17^8 + 11\cdot 17^9 + O(17^{10}),\\ &4 + 10\cdot 17^2 + 14\cdot 17^3 + 8\cdot 17^4 + 11\cdot 17^5 + 12\cdot 17^6 + 16\cdot 17^7 + 12\cdot 17^8 + 2\cdot 17^9 + O(17^{10}),\\ &4 + 15\cdot 17 + 16\cdot 17^3 + 16\cdot 17^4 + 4\cdot 17^5 + 15\cdot 17^6 + 8\cdot 17^7 + 5\cdot 17^8 + 16\cdot 17^9 + O(17^{10}),\\ &16 + 16\cdot 17 + 16\cdot 17^2 + 16\cdot 17^3 + 16\cdot 17^4 + 16\cdot 17^5 + 16\cdot 17^6 + 16\cdot 17^7 + 16\cdot 17^8 + 16\cdot 17^9 + O(17^{10}),\\ &0,\\ &0)\end{aligned}$$

Here is Sage code used to compute the dual basis:

```
R.<x> = QQ['x']
X = HyperellipticCurve((x^3+x+1)*(x^4+2*x^3-3*x^2+4*x+4))
g = X.genus()
prec=16
p = 17
K = Qp(p,prec)
from sage.schemes.hyperelliptic_curves.monsky_washnitzer import matrix_of_frobenius_hyperelliptic
N = X.change_ring(QQ).cup_product_matrix()
D = (N[[g..2*g-1],[0..g-1]])^-1
M,frob = matrix_of_frobenius_hyperelliptic(X.change_ring(K))
Mprec = M^prec
Mprecg = Mprec.columns()[g:2*g]
id = identity_matrix(2*g).columns()[0:g]
Atrans = matrix(2*g,2*g,[id[i] for i in range(g)] + [Mprecg[i] for i in range(g)])
A=Atrans.transpose()
Ainv = A^-1
Y = Ainv[[0..g-1],[g..2*g-1]]
wi_bar_vec = [0]*g
nDYt = -D*(Y.transpose())
for i in range(g):
    wi_bar_vec[i] = vector(K, [K(x) for x in nDYt.rows()[i].list()] + [K(x) for x in D.rows()[i].list(
```

Step (3): We give some details for the residue disk corresponding to $(3, 11) \in X(\mathbb{F}_{17})$, where we have the rational point $S = (3, 62)$. We have

$$\theta(S) = 11\cdot 17 + 17^2 + 16\cdot 17^3 + 3\cdot 17^4 + 4\cdot 17^5 + 12\cdot 17^6 + 15\cdot 17^7 + 6\cdot 17^8 + O(17^9),$$

which is computed as a local height. Using a local coordinate at z_P at P:

$$\begin{aligned}z_P = (&3 + z + O(z^{11}),\\ &62 + 65z + \frac{1587}{62}z^2 + \frac{15327}{3844}z^3 + \frac{70969}{476656}z^4 - \frac{3669}{953312}z^5 - \frac{295969}{1832265664}z^6 + \frac{28132747}{113600471168}z^7 - \frac{3499191357}{28172916849664}z^8 + O(z^9)),\end{aligned}$$

we compute

$$\begin{aligned}\theta(z) &:= \theta(P) - 2\left(\sum_{i=0}^{g-1}\int_P^{z_P}\omega_i\overline{\omega}_i + \sum_{i=0}^{g-1}\int_P^{z_P}\omega_i\int_b^P\overline{\omega}_i\right)\\
&= 11\cdot 17 + 7\cdot 17^2 + 17^3 + 16\cdot 17^4 + 3\cdot 17^5 + 6\cdot 17^6 + 2\cdot 17^7 + 6\cdot 17^8 + O(17^9) +\\
&\quad \left(4 + 10\cdot 17 + 17^2 + 10\cdot 17^3 + 16\cdot 17^4 + 7\cdot 17^5 + 17^6 + 5\cdot 17^7 + 9\cdot 17^8 + O(17^9)\right) z +\\
&\quad \left(4\cdot 17^{-1} + 5 + 4\cdot 17 + 3\cdot 17^2 + 8\cdot 17^3 + 3\cdot 17^4 + 13\cdot 17^5 + 3\cdot 17^6 + 6\cdot 17^7 + O(17^8)\right) z^2 +\\
&\quad \left(10\cdot 17^{-1} + 4 + 15\cdot 17 + 3\cdot 17^2 + 4\cdot 17^3 + 17^4 + 17^5 + 10\cdot 17^6 + 15\cdot 17^7 + O(17^8)\right) z^3 +\\
&\quad \left(11\cdot 17^{-1} + 1 + 17 + 9\cdot 17^2 + 4\cdot 17^4 + 5\cdot 17^5 + 14\cdot 17^6 + 2\cdot 17^7 + O(17^8)\right) z^4 +\\
&\quad \left(17^{-1} + 7 + 4\cdot 17 + 6\cdot 17^2 + 5\cdot 17^3 + 5\cdot 17^4 + 13\cdot 17^5 + 4\cdot 17^6 + 9\cdot 17^7 + O(17^8)\right) z^5 +\\
&\quad \left(13\cdot 17^{-1} + 14\cdot 17 + 17^2 + 10\cdot 17^3 + 16\cdot 17^4 + 4\cdot 17^5 + 15\cdot 17^6 + 13\cdot 17^7 + O(17^8)\right) z^6 +\\
&\quad \left(10\cdot 17^{-1} + 15 + 14\cdot 17 + 4\cdot 17^2 + 15\cdot 17^3 + 3\cdot 17^4 + 5\cdot 17^5 + 9\cdot 17^6 + 3\cdot 17^7 + O(17^8)\right) z^7 +\\
&\quad \left(7\cdot 17^{-1} + 5 + 11\cdot 17 + 16\cdot 17^2 + 6\cdot 17^3 + 3\cdot 17^4 + 5\cdot 17^5 + 8\cdot 17^7 + O(17^8)\right) z^8 +\\
&\quad \left(14\cdot 17^{-1} + 15 + 14\cdot 17 + 12\cdot 17^2 + 13\cdot 17^3 + 3\cdot 17^4 + 3\cdot 17^5 + 7\cdot 17^6 + 14\cdot 17^7 + O(17^8)\right) z^9 +\\
&\quad \left(11\cdot 17^{-1} + 12 + 5\cdot 17 + 2\cdot 17^2 + 10\cdot 17^3 + 4\cdot 17^5 + 5\cdot 17^6 + 6\cdot 17^7 + O(17^8)\right) z^{10} +\\
&\quad \left(11\cdot 17^{-1} + 4 + 13\cdot 17 + 13\cdot 17^2 + 12\cdot 17^3 + 2\cdot 17^4 + 15\cdot 17^5 + 6\cdot 17^6 + 10\cdot 17^7 + O(17^8)\right) z^{11} +\\
&\quad \left(7\cdot 17^{-1} + 16 + 16\cdot 17 + 10\cdot 17^2 + 15\cdot 17^3 + 9\cdot 17^4 + 4\cdot 17^6 + 9\cdot 17^7 + O(17^8)\right) z^{12} +\\
&\quad \left(13\cdot 17^{-1} + 3 + 17^2 + 4\cdot 17^3 + 14\cdot 17^4 + 7\cdot 17^5 + 13\cdot 17^6 + 2\cdot 17^7 + O(17^8)\right) z^{13} +\\
&\quad \left(6\cdot 17^{-1} + 1 + 13\cdot 17 + 4\cdot 17^2 + 6\cdot 17^3 + 13\cdot 17^4 + 11\cdot 17^5 + 12\cdot 17^6 + 15\cdot 17^7 + O(17^8)\right) z^{14} +\\
&\quad \left(16\cdot 17^{-1} + 15\cdot 17 + 11\cdot 17^2 + 10\cdot 17^3 + 14\cdot 17^4 + 16\cdot 17^5 + 6\cdot 17^6 + 4\cdot 17^7 + O(17^8)\right) z^{15} + O(z^{16}).\end{aligned}$$

Step (4): Recall from Example 2.39 that the set $\mathcal{S}$ in Theorem 2.34 is

$$\mathcal{S} = \left\{\{a\log_p 2 + b\log_p 31 : a \in \left\{0, 1, \frac{5}{4}, \frac{7}{4}\right\}, b \in \left\{0, \frac{1}{2}\right\}\right\}.$$

Step (5): We list the solutions modulo 17^3 that we found in the residue disk corresponding to $(3, 11) \in X(\mathbb{F}_{17})$, that we considered in Step (3). The integral point $S = (3, 62)$ in this disk satisfies $\rho(S) = \frac{1}{2}\log_p 31 + \frac{5}{4}\log_p 2$. There is another solution $z_1 = (3 + 16\cdot 17 + 8\cdot 17^2 + O(17^3), 11 + 6\cdot 17 + 2\cdot 17^2 + O(17^3))$ with the same ρ-value. The only other two solutions in this disk have ρ-value $\frac{1}{2}\log_p 31$; they are given by $z_2 = (3 + 12\cdot 17 + O(17^3), 11 + 17 + 4\cdot 17^2 + O(17^3))$ and $z_3 = (3 + 4\cdot 17 + 9\cdot 17^2 + O(17^3), 11 + 8\cdot 17 + 6\cdot 17^2 + O(17^3))$.

Step (6): In order to apply the Mordell–Weil sieve, we first need to compute the torsion subgroup of $J(\mathbb{Q})$. There is an obvious point $T \in J(\mathbb{Q})$ of order 2. `Magma's TorsionBound`, which uses reduction modulo a number of good primes to compute an upper bound on $\#J(\mathbb{Q})_{\mathrm{tors}}$, returns 4. One may show that $J(\mathbb{Q}) \simeq \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$ by proving that T is not divisible by 2 in $J(\mathbb{Q})$, again using reduction. A general algorithm to compute the torsion subgroup of $J(\mathbb{Q})$ for Jacobians of hyperelliptic genus 3 curves will be given in forthcoming work of Müller and Reitsma; for genus 2, one can use `Magma's TorsionSubgroup`, due to Stoll [Sto99].

The Mordell–Weil sieve requires primes v of good reduction such that $17 \mid \#J(\mathbb{F}_v) > 0$. We find that for $v < 20{,}000$, the largest 17-adic valuation of the exponent of $J(\mathbb{F}_v)$ is 2. This suggests using the modulus $M = 17^2$. For instance, consider the 17-adic approximate solution z_1 and suppose that there

is a rational point $P \in X(\mathbb{Q})$ such that P reduces to z modulo 17^2. Linearity of Coleman integrals of the holomorphic differentials $\frac{x^i dx}{2y}$ for $i = 0, 1, 2$ implies that we have

$$[(P) - (\infty)] = a_1 P_1 + a_2 P_2 + a_3 P_3 + a_4 T\,,$$

where

$$a_1 \equiv 13 + 9 \cdot 17 \bmod 17^2\,, \quad a_2 \equiv 8 + 14 \cdot 17 \bmod 17^2\,, \quad a_3 \equiv 7 + 9 \cdot 17 \bmod 17^2\,, \quad a_4 \in \{0, 1\}\,. \tag{23}$$

We want to show that no such point P can exist. To this end, we use the prime $v = 617$. The two putative tuples $(\tilde{a}_1, \ldots, \tilde{a}_4) \in (\mathbb{Z}/17^2\,\mathbb{Z})^4$ from (23) give rise to two cosets in $J(\mathbb{F}_{617})/17^2$, and neither of these meet the image of $X(\mathbb{F}_{617})$. The same is true for the solutions z_2 and z_3, so we find that S is the only integral point in its residue disk. How did we come up with the choice $v = 617$? The reason is that $J(\mathbb{F}_{617}) \simeq (\mathbb{Z}/2\,\mathbb{Z})^3 \times (\mathbb{Z}/(3^2 \cdot 17 \cdot 2)\,\mathbb{Z})^3$, so it's reasonable to expect that it leads to nontrivial information for the modulus $M = 17^2$. The same is true, for instance, for $v = 607$, since $J(\mathbb{F}_{607}) \simeq (\mathbb{Z}/2\,\mathbb{Z})^4 \times \mathbb{Z}/4\,\mathbb{Z} \times \mathbb{Z}/(17^2 \cdot 12304)\,\mathbb{Z}$. Indeed, we can use $v = 607$ to show that z_2 and z_3 do not come from a rational point, but this does not work for z_1.

After running the Mordell-Weil sieve for $v = 607$ and $v = 617$ with the modulus $M = 17^2$, we find that only 2 cosets are left which do not seem to come from rational points, but for which these choices of v cannot prove that this is true. Unfortunately, using all good primes $v < 20,000$ such that $17 \mid \#J(\mathbb{F}_v) > 0$ does not improve this situation. Instead of increasing the size of the primes v further, which becomes quite expensive, we change the modulus M to $M = 3 \cdot 17^2$. Applying the Chinese Remainder Theorem we obtain a list of putative coefficient tuples $(\tilde{a}_1, \ldots, \tilde{a}_4) \in (\mathbb{Z}/3 \cdot 17^2\,\mathbb{Z})^4$

Working with the primes $v \in \{607, 617, 11131\}$ and the prime $v = 11131$, which satisfies , $J(\mathbb{F}_{11131}) \simeq \mathbb{Z}/3\,\mathbb{Z} \times \mathbb{Z}/(2 \cdot 3)\,\mathbb{Z} \times \mathbb{Z}/(3 \cdot 17^2 \cdot 37 \cdot 2340692)\,\mathbb{Z}$, we find that none of the cosets arising from a solution outside the known integral points comes from a rational point.[12]

This completes the proof that

$$\mathscr{U}(\mathbb{Z}) = \{(-1, \pm 2)\,,\ (0, \pm 2)\,,\ (-2, \pm 12)\,,\ (3, \pm 62)\}\,.$$

3. Nekovář's p-adic heights

The quadratic Chabauty method uses p-adic heights to cut out rational points on certain nice curves $X/\mathbb{Q}$ via bilinear relations. We showed in §2.3 that the construction of Coleman and Gross leads to an algorithm to compute the integral points on certain hyperelliptic curves, and we mentioned why this approach does not extend easily to rational points.

As explained below in Section 4, we will in fact cut out a subset $X(\mathbb{Q}_p)_U \supseteq X(\mathbb{Q})$ of $X(\mathbb{Q}_p)$, depending on a certain non-abelian unipotent quotient U of the $\mathbb{Q}_p$-étale fundamental group of $X_{\overline{\mathbb{Q}}}$. The set $X(\mathbb{Q}_p)_U$ is defined using a non-abelian generalization of Chabauty's method due to Kim, which is discussed in the contribution of Kim and Lüdtke in this volume and which we briefly summarize i §4.1. Kim's philosophy suggests that our construction should not use the geometry of the Jacobian, but rather a "motivic" version (but see Remark 5.7 below). In this section, we recall a motivic construction of p-adic heights, due to Nekovář [Nek93].

[12] In [BBM17, Example 9.2], we also used quadratic Chabauty for the primes $p = 7$ and $p = 37$, but this is actually not required.

3.1. **p-adic Hodge theory.** We begin by briefly recalling some notions from p-adic Hodge theory. A good reference for most of what we need in this section is [Bel09], but [Nek93, §1] recalls the relevant background in a concise way. See [Ber04, And03, BC09] for other accounts of p-adic Hodge theory. In §4.1 we will use, in addition, results from Olsson's non-abelian p-adic Hodge theory [Ols11].

Fix a prime p. For any prime v, let G_v denote the absolute Galois group of $\mathbb{Q}_v$. Let V be a p-adic Galois representation, by which we mean a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous action of G_p.

The starting point of p-adic Hodge theory is the observation that in contrast to ℓ-adic representations, for p-adic Galois representations the condition of being unramified is too restrictive to be useful. To remedy this, Fontaine defined a p-adic period ring B_{cris} with a Frobenius action, and a functor D_{cris} by

$$D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_p}.$$

We always have

$$\dim_{\mathbb{Q}_p} D_{\mathrm{cris}}(V) \leq \dim_{\mathbb{Q}_p}(V).$$

If equality holds, then we say that V is *crystalline*. If V is unramified, then it is crystalline.

Example 3.1. To see an indication why this is the right analogue for $\ell = p$ of unramifiedness for $\ell \neq p$, recall that by the Néron–Ogg–Shafarevich criterion, an abelian variety $A/\mathbb{Q}_p$ has good reduction if and only if the ℓ-adic Tate module $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim A[\ell^n]$ is an unramified representation. This criterion does not extend to $\ell = p$. Instead, one can show that A has good reduction if and only if the p-adic Tate module is crystalline.

In particular, for a nice curve $X/\mathbb{Q}$ of good reduction at p, we have $D_{\mathrm{cris}}(\mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)) \simeq \mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})$, and it turns out that $\mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ is crystalline (see [Fal89]). The Frobenius action on B_{cris} induces a Frobenius action on $D_{\mathrm{cris}}(\mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p))$; this is the same as the Frobenius action on $\mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})$ coming from crystalline cohomology (for the construction of the Frobenius action in terms of rigid cohomology, see the more general version of Theorem 1.27 due to Baldassarri–Chiarellotto [BC94]).

When G is a topological group and V_1, V_2 are finite-dimensional continuous $\mathbb{Q}_p$-representations of G, we can identify $\mathrm{H}^1(G, V_1^* \otimes V_2)$ with the group $\mathrm{Ext}^1(V_1, V_2)$. Hence, an element $\xi \in \mathrm{H}^1(G_p, V)$ corresponds to an isomorphism class of extension of $\mathbb{Q}_p$ by V

$$0 \to V \to E \to \mathbb{Q}_p \to 0;$$

here ξ is the image of the neutral element of $\mathrm{H}^0(G_p, \mathbb{Q}_p)$ under the connecting homomorphism $\mathrm{H}^0(G_p, \mathbb{Q}_p) \to \mathrm{H}^1(G_p, V)$. We call ξ *crystalline*, provided that the Galois representation E is.

Definition 3.2. The *local Bloch–Kato Selmer group* $\mathrm{H}^1_f(G_p, V)$[13] is the set of crystalline classes in $\mathrm{H}^1(G_p, V)$. The (global) *Bloch–Kato Selmer group* $\mathrm{H}^1_f(G_{\mathbb{Q}}, V)$ is the group of $\xi \in \mathrm{H}^1(G_{\mathbb{Q}}, V)$ whose image $\mathrm{loc}_v(V) \in \mathrm{H}^1(G_v, V)$ is crystalline for $v = p$ and unramified for all $v \neq p$.

Example 3.3. Suppose that K is a finite extension of $\mathbb{Q}_p$. Then Kummer theory gives an isomorphism

$$\kappa\colon \widehat{K^*} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{H}^1(G_K, \mathbb{Q}_p(1)),$$

where $\widehat{K^*} = \varprojlim K^* \otimes \mathbb{Z}/p^n\mathbb{Z}$ is the p-adic completion and $\mathbb{Q}_p(1)$ is the one-dimensional representation given by the p-adic cyclotomic character $\chi_p\colon G_p \to \mathbb{Z}_p^\times$; see [BC09, §1.1]. According to [Bel09, Proposition 2.9], this isomorphism identifies $\mathscr{O}_K^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $\mathrm{H}^1_f(G_p, \mathbb{Q}_p(1))$.

[13] The f stands for "finite"

Example 3.4. Let K be a number field. Then, by [Bel09, Proposition 2.12] we have

$$\mathrm{H}^1_f(G_K, \mathbb{Q}_p(1)) \simeq \mathscr{O}_K^* \otimes_{\mathbb{Z}} \mathbb{Q}_p\,.$$

Remark 3.5. More generally, for a topological group G and finite-dimensional continuous $\mathbb{Q}_p$-representations V_1, V_2 of G, we can define and identify $\mathrm{H}^1_f(G, V_1^* \otimes V_2)$ and $\mathrm{Ext}^1_f(V_1, V_2)$, where the former contains crystalline torsors and the latter contains crystalline extensions.

Nekovář's approach is inspired by a motivic construction due to Scholl [Sch94] of archimedean local height pairings arising in Beilinson's extension of canonical heights to Chow groups based on mixed Hodge structures.

3.2. **Nekovář's construction of p-adic heights.** Fix a prime p and a finite set of primes T_0. Let $T := T_0 \cup \{p\}$. For "good"[14] p-adic Galois representations V, Nekovář constructs a bilinear p-adic height pairing on Bloch–Kato Selmer groups

$$h\colon\ \mathrm{H}^1_f(G_{\mathbb{Q}}, V) \times \mathrm{H}^1_f(G_{\mathbb{Q}}, V^*(1)) \to \mathbb{Q}_p\,.$$

This global p-adic height depends only on

(a) the choice of an idèle class character $\chi\colon \mathbb{A}_{\mathbb{Q}}^{\times}/\,\mathbb{Q}^{\times} \to \mathbb{Q}_p$,
(b) a splitting s of the Hodge filtration on $V_{\mathrm{dR}} := D_{\mathrm{cris}}(V)$.

For everything that follows, we let $X/\,\mathbb{Q}$ denote a nice curve of genus $g \geq 2$ such that X has good reduction at p and such that T_0 contains the set of primes of bad reduction for X. We set $V = \mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}})^*$, This V is "good" in the sense of Nekovář; in particular V is crystalline and $V_{\mathrm{dR}} = \mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})^*$ by a theorem of Faltings [Fal89]. The choices (a) and (b) are exactly the choices required in the construction of Coleman and Gross.

In [Nek93, Section 2], Nekovář presents a global construction of the height pairing h. We will not discuss it here, but rather focus on a construction of local heights h_v, so that we have

$$h = h_p + \sum_{v \neq p} h_v.$$

See [Nek93, §4] for more details. See also [BDM+19, §3], [BD18, §4.2] for similar treatments, and a reformulation in terms of non-abelian cohomology. A generalization of Nekovář's construction is discussed in [BD21].

Recall that, starting with two points in $J(\mathbb{Q})$, the local Coleman–Gross heights are defined by first choosing divisors of degree 0 representing these points. The local heights then depend on these choices, whereas the global height does not. In our present setting, the idea is to interpret classes $e_1 \in \mathrm{H}^1_f(G_{\mathbb{Q}}, V)$ and $e_2 \in \mathrm{H}^1_f(G_{\mathbb{Q}}, V^*(1))$ as extensions

$$0 \to V \to E_1 \to \mathbb{Q}_p \to 0$$
$$0 \to \mathbb{Q}_p(1) \to E_2 \to V \to 0,$$

where we identify $\mathrm{H}^1_f(G_{\mathbb{Q}}, V)$ with crystalline extensions of V by $\mathbb{Q}_p$ with a continuous $G_{\mathbb{Q}}$-action. Nekovář shows [Nek93, Proposition 4.4] that one can lift these extensions to form a *mixed extension* of E_1 and E_2: a p-adic $G_{\mathbb{Q}}$-representation E with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$, sitting in a commutative diagram

[14] The conditions for being "good" are spelled out in [Nek93, 2.1.2]. In particular, we want V to be crystalline at p and unramified outside T.

(24)
$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & E_2 & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & E & \longrightarrow & E_1 & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathbb{Q}_p & \xrightarrow{=} & \mathbb{Q}_p & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} .$$

and having a weight filtration by $G_{\mathbb{Q}}$-subrepresentations

$$0 = W_{-3}E \subseteq W_{-2}E \subseteq W_{-1}E \subseteq W_0E = E\,,$$

such that $W_{-1}E \simeq E_2$ and $W_0E/W_{-2}E \simeq E_1$. In other words, the mixed extension E has a block matrix representation

$$\begin{pmatrix} 1 & 0 & 0 \\ * & \rho_V & 0 \\ * & * & \chi_p \end{pmatrix}$$

where the representation ρ_V corresponds to V and χ_p is the p-adic cyclotomic character.

Given any mixed extension E of $G_{\mathbb{Q}}$-representations with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$, we can define projections

$$\pi_1(E) := [W_0E/W_{-2}E] \in \mathrm{H}^1(G_{\mathbb{Q}}, V), \quad \pi_2(E) := [W_{-1}E] \in \mathrm{H}^1(G_{\mathbb{Q}}, V^*(1))\,. \tag{25}$$

For a mixed extension E of E_1 and E_2 as above, we then have $\pi_i(E) = e_i$.

For every prime v, we can define local mixed extension of G_v-representations with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$ in an analogous way. We say a global mixed extension E is *crystalline* if $\mathrm{loc}_p(E)$ is crystalline. If E is crystalline, the projections $\pi_i(E)$ are crystalline as well.

Nekovář defines a local height h_v on mixed extension of G_v-representations with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$. We will assume it is of the form $\mathrm{loc}_v(E)$ for a global mixed extension E. The local height is not well-defined on $\mathrm{H}^1_f(G_v, V) \times \mathrm{H}^1_f(G_v, V^*(1))$, it depends on the chosen mixed extension (in fact on its equivalence class). Such mixed extensions can be added via the Baer sum; the local heights are then bi-additive in the sense of [BD18, Definition 4.4]. By [Nek93, Theorem 4.11],

$$h(e_1, e_2) = \sum_v h_v(\mathrm{loc}_v(E)) \tag{26}$$

is independent of the choice of E and defines a bilinear pairing

$$h\colon \mathrm{H}^1_f(G_{\mathbb{Q}}, V) \times \mathrm{H}^1_f(G_{\mathbb{Q}}, V^*(1)) \to \mathbb{Q}_p\,.$$

By Poincaré duality, we have $V \simeq V^*(1)$, so we in fact get a bilinear pairing

$$h\colon \mathrm{H}^1_f(G_{\mathbb{Q}}, V) \times \mathrm{H}^1_f(G_{\mathbb{Q}}, V) \to \mathbb{Q}_p\,.$$

Remark 3.6. If $\ker(s)$ is isotropic with respect to the dual of the cup product, then this pairing is symmetric by [Nek93, Theorem 4.11 (4)].

Remark 3.7. One can associate a mixed extension as above to a pair of divisors $D_1, D_2 \in \mathrm{Div}^0(X)$ with disjoint support via an étale Abel-Jacobi map, see [Nek93, Section 5]. Besser [Bes04] shows that for our choice of V, the local Coleman–Gross and Nekovář heights (with respect to these choices) are equivalent. For some examples, this equivalence will be crucial to run quadratic Chabauty in practice, see §5.3.2 and §5.5 below.

3.3. **Local heights.** The construction of the local heights h_v is not particularly intuitive. The rough idea is to construct a class $c \in \mathrm{H}^1(G_v, \mathbb{Q}_p(1))$ (crystalline when $v = p$) from a local mixed extension E_v. One can then use the Kummer isomorphism

$$\kappa_v \colon \widehat{\mathbb{Q}_v^*} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{H}^1(G_v, \mathbb{Q}_p(1))$$

to define a p-adic number

$$h_v(E_v) := \chi_v(c) \in \mathbb{Q}_p, \tag{27}$$

where χ_v is the map

$$\chi_v \colon \mathrm{H}^1(G_v, \mathbb{Q}_p(1)) \to \widehat{\mathbb{Q}_v^*} \otimes \mathbb{Q}_p \to \mathbb{Q}_p \tag{28}$$

induced by the local component $\chi_v \colon \mathbb{Q}_v^* \to \mathbb{Q}_p$ of our chosen idèle class character and by κ_v^{-1}. Our exposition follows [BDM+19, §3].

3.3.1. *Local heights away from p.* First consider a prime $\ell \neq p$. Our main focus will be on algorithms for h_p, so we will only discuss this case briefly. Not that $\chi_\ell(\mathbb{Z}_\ell^*) = 0$ because of continuity, hence the second map in (28) factors through the valuation $\mathrm{ord}_\ell \colon \mathbb{Q}_\ell^* \to \mathbb{Z}$ for $v = \ell$.

It is explained in [BDM+19, §3.2] that we have $\mathrm{H}^1(G_\ell, V) = \mathrm{H}^1(G_\ell, V^*(1)) = 0$, essentially since (by the weight-monodromy conjecture for curves and Proposition A.2, the six-term exact sequence in non-abelian Galois cohomology) $\mathrm{H}^0(G_\ell, V) = \mathrm{H}^0(G_\ell, V^*(1))^* = 0$. Hence, if E_ℓ is a mixed extension of G_ℓ-representations with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$, then, from the local version of the diagram (24), we obtain a splitting $E_\ell \simeq V \oplus N$, where N is an extension

$$0 \to \mathbb{Q}_p(1) \to N \to \mathbb{Q}_p \to 0\,,$$

so the class $c := [N]$ lies in $\mathrm{H}^1(G_\ell, \mathbb{Q}_p(1))$, and we define $h_\ell(E_\ell)$ as in (27). See [Nek93, §4.6] and [BDM+19, §3.2]. If E_ℓ is unramified, then $h_\ell(E_\ell) = 0$, so the sum in (26) is finite. More generally, a simple argument shows (see [BDM+19, Lemma 3.2]):

Remark 3.8. Suppose that E_ℓ is potentially unramified. Then $h_\ell(E_\ell)$ is trivial. This implies that the local heights at ℓ of interest to us are trivial when X has potentially good reduction at ℓ.

3.3.2. *Local heights at p.* We now describe the main object we will need to compute in order to apply quadratic Chabauty for rational points: the local height $h_p(E_p)$, where E_p is a crystalline mixed extension E_p of G_p-extensions with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$. The construction is in terms of p-adic Hodge theory. More precisely, the local height $h_p(E_p)$ is defined in terms of $D_{\mathrm{cris}}(E_p)$, which turns out to be a mixed extension of filtered ϕ-module, defined below. For the mixed extensions of interest to us, we will show later that we can make the relevant structure of $D_{\mathrm{cris}}(E_p)$ explicit.

The definition of $h_p(E_p)$ is similar to the construction of local heights away from p, but the construction of the class $c \in \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1))$ is more involved, because we do not have $\mathrm{H}^1(G_p, V) = \mathrm{H}^1(G_p, V^*(1)) = 0$. We will end this section by making the construction rather explicit, in terms of splittings of filtered ϕ-modules.

Definition 3.9. A *filtered ϕ-module* (over $\mathbb{Q}_p$) is a finite dimensional $\mathbb{Q}_p$-vector space W, equipped with an exhaustive and separated decreasing filtration Fil^i and an automorphism ϕ. Recall that

- exhaustive means $W = \cup_i \mathrm{Fil}^i$,
- separated means $\cap_i \mathrm{Fil}^i = 0$,
- decreasing means $\mathrm{Fil}^{i+1} \subseteq \mathrm{Fil}^i$.

Example 3.10. Here are some examples of filtered ϕ-modules:

(1) $\mathbb{Q}_p$ with $\mathrm{Fil}^0 = \mathbb{Q}_p$, $\mathrm{Fil}^n = 0$ for all $n > 0$, and $\phi = id$.
(2) $\mathbb{Q}_p(1) = D_{\mathrm{cris}}(\mathbb{Q}_p(1))$ with $\mathrm{Fil}^{-1} = \mathbb{Q}_p$, $\mathrm{Fil}^n = 0$ for all $n > -1$, and $\phi = 1/p$.
(3) Recall from Example 3.1 that $\mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ is crystalline and that $\mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p}) = D_{\mathrm{cris}}(\mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p))$. The Frobenius action from crystalline cohomology and the Hodge filtration endow $\mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})$ with the structure of a filtered ϕ-module.
(4) $V_{\mathrm{dR}} := \mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})^* = D_{\mathrm{cris}}(V)$ with the dual filtration and action.
(5) The trivial mixed extension $\mathbb{Q}_p \oplus V \oplus \mathbb{Q}_p(1)$ with filtration and automorphism induced by those on its graded pieces.

In general, we think of the filtration as a Hodge filtration and of the automorphism as a Frobenius action coming from comparison theorems.

Remark 3.11. To be precise, all filtered ϕ-modules below are *admissible* (i.e. they come from p-adic Galois representations), but we drop the adjective for simplicity.

To define Nekovar's local height, we work with extensions of filtered ϕ-modules. For two filtered ϕ-modules W_1 and W_2, we let $\mathrm{Ext}^1_{\mathrm{Fil},\phi}(W_1, W_2)$ denote the $\mathbb{Q}_p$-vector space of extensions

$$0 \to W_2 \to E \to W_1 \to 0$$

of filtered ϕ-modules.

Suppose that W is a filtered ϕ-module satisfying (31) and let $[E] \in \mathrm{Ext}^1_{\mathrm{Fil},\phi}(\mathbb{Q}_p, W)$, i.e.

$$0 \to W \to E \to \mathbb{Q}_p \to 0 \tag{29}$$

is an extension of $\mathbb{Q}_p$ by W in the category of filtered ϕ-modules. We now explain how to associate an element of W/F^0 to E.

If $s, s' \colon \mathbb{Q}_p \to E$ are splittings of (29), then we have for any $a \in \mathbb{Q}_p$

$$s(a) - s'(a) \in \ker(E \to \mathbb{Q}_p),$$

so that by exactness there is a unique element $w(s, s') \in W$ mapping to $s(1) - s'(1)$. In particular, if s and s' are both ϕ-equivariant, then, $w(s, s')$ is fixed by ϕ, since ϕ is the identity on $\mathbb{Q}_p$. By (29), we have $s = s'$. Hence there is a unique ϕ-equivariant splitting, which we denote by s^ϕ.

Now suppose that s and s' respect the Hodge filtration, by which we mean that the induced decompositions $E \simeq s(\mathbb{Q}_p) \oplus W$ ($E \simeq s'(\mathbb{Q}_p) \oplus W$, respectively) respect the Hodge filtrations. Since $F^0 \mathbb{Q}_p = \mathbb{Q}_p$, we get $w(s, s') \in F^0 W$. In fact, if s and s' respects the Hodge filtration, then s' does too if and only if $w(s, s') \in F^0 W$. By choosing such a splitting s^H, we get an injection

$$\epsilon \colon \mathrm{Ext}^1_{\mathrm{Fil},\phi}(\mathbb{Q}_p, W) \to W/\mathrm{Fil}^0 ; \quad E \mapsto w(s^\phi, s^H) . \tag{30}$$

In fact, this injection is an isomorphism.

Remark 3.12. The fact that ϵ is an isomorphism is similar to the invertibility of the Bloch–Kato exponential, see [BK90, §3.8]. This is a surjective homomorphism $D_{\mathrm{dR}}(U)/F^0 \to H^1_f(G_p, U)$, where

$D_{\mathrm{dR}}(U) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_p}$ and $B_{\mathrm{dR}} \supset B_{\mathrm{cris}}$ is another period ring due to Fontaine, which carries a Hodge filtration. Here U is a *de Rham* representation, which means that it satisfies $\dim_{\mathbb{Q}_p} D_{\mathrm{dR}}(U) = \dim_{\mathbb{Q}_p} U$. A crystalline representation is de Rham. Bloch–Kato show that when

$$D_{\mathrm{cris}}(U)^{\phi=1} = \{0\}, \tag{31}$$

then the Bloch–Kato exponential has an inverse, the *Bloch-Kato logarithm*. By the Weil conjectures, (31) is satisfied for $U = V = \mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}})^*$. In this case, the Bloch-Kato logarithm provides an intrinsic way to define the abelian logarithm on the Jacobian of $X_{\mathbb{Q}_p}$. This is crucial for non-abelian Chabauty; see for instance [Cor19, §3.2.3].

The filtered ϕ-module $\mathbb{Q}_p \oplus V \oplus \mathbb{Q}_p(1)$ has a weight filtration, just like the mixed extensions of Galois representations we encountered above. We call such objects *mixed extensions of filtered ϕ-modules with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$*.

Now let E_p be a crystalline mixed extension of G_p-representations with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$. Then $E_{\mathrm{dR}} := D_{\mathrm{cris}}(E_p)$ is a mixed extension of filtered ϕ-modules with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$. In analogy with the case of Galois-representations, we can define extensions

$$E_1 := E_{\mathrm{dR}}/\,\mathbb{Q}_p(1) \quad \text{and} \quad E_2 := \ker(E_{\mathrm{dR}} \to \mathbb{Q}_p)$$

of filtered ϕ-modules:

$$0 \to V_{\mathrm{dR}} \to E_1 \to \mathbb{Q}_p \to 0 \tag{32}$$

and

$$0 \to \mathbb{Q}_p(1) \to E_2 \to V_{\mathrm{dR}} \to 0, \tag{33}$$

fitting into a commutative diagram (24) of filtered ϕ-modules.

Recall from (27) that we want to define the local height in terms of an element $c \in H^1_f(G_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p(1)$. To construct c, we use the exact sequence

$$0 \to \mathbb{Q}_p(1) \to E_2/F^0 \xrightarrow{\pi} V_{\mathrm{dR}}/F^0 \to 0 \tag{34}$$

induced by (33). Therefore it suffices to construct two elements of E_2 with the same image under π.

One such element is given by viewing E_{dR} as an extension

$$0 \to E_2 \to E_{\mathrm{dR}} \to \mathbb{Q}_p \to 0$$

this corresponds to an element $e = \epsilon([E_{\mathrm{dR}}]) \in E_2/F^0$.

As above, the exact sequence

$$0 \to \mathbb{Q}_p(1) \to E_2 \to V_{\mathrm{dR}} \to 0.$$

admits a unique ϕ-equivariant splitting $\gamma^\phi \colon V_{\mathrm{dR}} \to E_2$. Recall that we have chosen a splitting $s \colon V_{\mathrm{dR}}/\operatorname{Fil}^0 \to V_{\mathrm{dR}}$ of the Hodge filtration on V_{dR}. We then apply the composition

$$\delta \colon V_{\mathrm{dR}}/\operatorname{Fil}^0 \xrightarrow{s} V_{\mathrm{dR}} \xrightarrow{\gamma^\phi} E_2 \to E_2/\operatorname{Fil}^0$$

to define

$$e' := \delta \circ \pi(e)\,.$$

Then both $\pi(e)$ and $\pi(e')$ are equal to $\epsilon([E_1])$ (defined using (32)), and so we have

$$\pi(e) - \pi(e') = 0 \in V_{\mathrm{dR}}/\operatorname{Fil}^0\,.$$

Hence we obtain

$$c := e - e' \in \mathbb{Q}_p(1) \simeq \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)). \tag{35}$$

by (34) and by Kummer theory.

Finally, we define the height of E_p by

$$h_p(E_p) := \chi_p(c). \tag{36}$$

In order to apply quadratic Chabauty in practice, it will be crucial to compute (36). To this end, we now give a more explicit version. By fixing a vector space splitting

$$s_0 \colon \mathbb{Q}_p \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}_p(1) \xrightarrow{\sim} E_{\mathrm{dR}}.$$

we may view E_{dR} as $\mathbb{Q}_p \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}_p(1)$.

As above, there is a unique ϕ-equivariant splitting s^ϕ

$$s^\phi \colon \mathbb{Q}_p \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}_p(1) \xrightarrow{\sim} E_{\mathrm{dR}},,$$

whereas a splitting

$$s^{\mathrm{Fil}} \colon \mathbb{Q}_p \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}_p(1) \xrightarrow{\sim} E_{\mathrm{dR}},,$$

which respects the Hodge filtration is only unique up to an element of E_2/Fil^0. Here the Hodge filtration on $\mathbb{Q}_p \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}_p(1)$ is given by

$$\begin{array}{lcl} \operatorname{Fil}^{-1} & = & (\mathbb{Q}_p \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}_p(1) \\ \operatorname{Fil}^{0} & = & \mathbb{Q}_p \oplus \operatorname{Fil}^0 V_{\mathrm{dR}} \\ \operatorname{Fil}^{1} & = & 0. \end{array} \tag{37}$$

Suppose that we have chosen bases for $\mathbb{Q}_p, V_{\mathrm{dR}}, \mathbb{Q}_p(1)$ such that with respect to these bases we have

$$s_0^{-1} \circ s^\phi = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_\phi & 1 & 0 \\ \gamma_\phi & \beta_\phi^{\intercal} & 1 \end{pmatrix}, \qquad s_0^{-1} \circ s^{\mathrm{Fil}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma_{\mathrm{Fil}} & \beta_{\mathrm{Fil}}^{\intercal} & 1 \end{pmatrix}$$

Here the non-uniqueness of s^{Fil} allows us to find an s^{Fil} and bases such that the "α_{Fil}-term" in $s_0^{-1} \circ s^{\mathrm{Fil}}$ vanishes. This determines s^{Fil} uniquely, which will be important in finding an explicit expression for it in Algorithm 5.20 below.

Finally, the splitting $s \colon V_{\mathrm{dR}}/\operatorname{Fil}^0 \to V_{\mathrm{dR}}$ of the Hodge filtration defines projections

$$s_1,\, s_2 \colon V_{\mathrm{dR}} \to s(V_{\mathrm{dR}}/\operatorname{Fil}^0) \oplus \operatorname{Fil}^0 \simeq V_{\mathrm{dR}}$$

Then we obtain the following expression for $h_p(E_p)$, which turns out to be suitable for explicit computations.

Proposition 3.13. *We have*

$$h_p(E_p) = \chi_p(\gamma_\phi - \gamma_{\mathrm{Fil}} - \beta_\phi^{\intercal} s_1(\alpha_\phi) - \beta_{\mathrm{Fil}}^{\intercal} s_2(\alpha_\phi)).$$

Proof. Recall from (35) that

$$h_p(E_p) = \chi_p(e - e'),$$

and that we have chosen a vector space splitting s_0. We first describe $e = \epsilon([E_{\mathrm{dR}}])$ in terms of s^ϕ and s^{Fil}. One can show that the map $1 \mapsto (1, \alpha_\phi, \gamma_\phi)$ defines the ϕ-equivariant splitting $\mathbb{Q}_p \to E_{\mathrm{dR}}$ of the exact sequence

$$0 \to E_2 \to E_{\mathrm{dR}} \to \mathbb{Q}_p \to 0\,. \tag{38}$$

and that a choice of a Hodge-filtration respecting splitting is given by $1 \mapsto (1, 0, \gamma_{\mathrm{Fil}})$. This yields

$$e = \epsilon([E_{\mathrm{dR}}]) = (\alpha_\phi, \gamma_\phi - \gamma_{\mathrm{Fil}}) + \operatorname{Fil}^0 E_2\,. \tag{39}$$

Futhermore, we have

$$e' = \delta(\pi(e)) = (s_1(\alpha_\phi), \beta_{\mathrm{Fil}}^{\mathsf{T}} \cdot s_1(\alpha_\phi) + \mathrm{Fil}^0 E_2\,. \tag{40}$$

Since $(v, \beta_{\mathrm{Fil}}^{\mathsf{T}} v) \in \mathrm{Fil}^0 E_2$ for any $v \in \mathrm{Fil}^0 V_{\mathrm{dR}}$, we find

$$\begin{aligned} e - e' &= (s_2(\alpha_\phi), \gamma_\phi - \gamma_{\mathrm{Fil}} - \beta_{\mathrm{Fil}}^{\mathsf{T}} s_1(\alpha_\phi)) + \mathrm{Fil}^0 E_2 \\ &= (0, \gamma_\phi - \gamma_{\mathrm{Fil}} - \beta_{\mathrm{Fil}}^{\mathsf{T}} s_1(\alpha_\phi) - \beta_{\mathrm{Fil}}^{\mathsf{T}} s_2(\alpha_\phi)) + \mathrm{Fil}^0 E_2\,. \end{aligned}$$

□

4. Quadratic Chabauty: theory

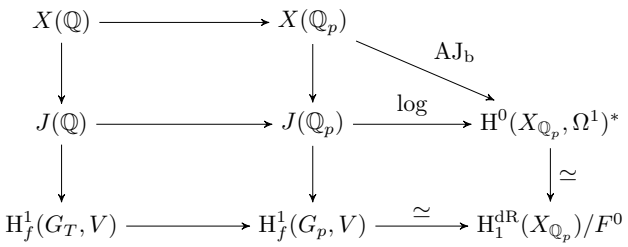
In this section we discuss the theoretical justification for the quadratic Chabauty method. Since we focus on computational methods in this course and the foundations of non-abelian Chabauty, of which quadratic Chabauty is a special case, are covered in Kim's lectures, we will be brief. Kim's approach relies on choosing a unipotent quotient U of the $\mathbb{Q}_p$-étale fundamental group of a curve and defining a corresponding subset of p-adic points containing the rational points using local conditions. The hope is that this set can be proved to be finite and can be computed explicitly.

Our main situation of interest is when the Chabauty condition is not satisfied, but the curve satisfies the quadratic Chabauty condition (44). In particular, this condition holds when the Mordell–Weil rank is equal to the genus and the Picard number is greater than 1, a situation frequently encountered for modular curves, as shown by Siksek [Sik17]. In this case, we can construct a non-abelian quotient U such that the corresponding set of p-adic points is finite and we can often indeed compute it. In the present section, we show finiteness, following [BD18].

4.1. **Chabauty–Kim theory.** Let X be a nice curve over $\mathbb{Q}$ of genus $g > 1$, and let J be its Jacobian. Assume that $X(\mathbb{Q}) \neq \varnothing$ and fix $b \in X(\mathbb{Q})$. We also fix a prime p of good reduction, we let T_0 denote the set of bad primes for X, and we set $T := T_0 \cup \{p\}$.

We begin by reformulating classical Chabauty in terms of p-adic Hodge theory, see also [Cor19] and Zureick-Brown's lectures. As in the previous section, we let $V := \mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)^*$, and $V_{\mathrm{dR}} := \mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})^*$, viewed as a filtered vector space with the dual filtration to the Hodge filtration, so that there is an isomorphism $V_{\mathrm{dR}}/\mathrm{Fil}^0 \simeq \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$. Let G_T be the maximal quotient of $G_{\mathbb{Q}}$ unramified outside T. The étale formulation of classical Chabauty can be summarized in the following commutative diagram:

(41)



We give a very brief summary of Kim's generalization, referring to [Kim09] and Kim's lectures for more details. Choose a Galois-stable unipotent quotient U of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}})_{\mathbb{Q}_p}$, the unipotent $\mathbb{Q}_p$-étale fundamental group of $X_{\overline{\mathbb{Q}}}$ with base point b. The latter is the $\mathbb{Q}_p$-pro-unipotent completion of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}})$. We also want U to be motivic, in the sense that it also has a de Rham realization.

There is a commutative diagram

$$\begin{array}{ccc} X(\mathbb{Q}) & \longrightarrow & \prod_{v\in T} X(\mathbb{Q}_v) \\ \downarrow{\scriptstyle j_U} & & \downarrow{\scriptstyle \prod j_{U,v}} \\ \mathrm{H}^1(G_T, U) & \xrightarrow{\prod \mathrm{loc}_v} & \prod_{v\in T} \mathrm{H}^1(G_v, U). \end{array}$$

where j_U and $j_{U,v}$ denote the global, respectively local, unipotent Kummer maps defined in [Kim05, Kim09]. It is a highly nontrivial result due to Kim [Kim05, Kim09] that the nonabelian pointed continuous cohomology sets $\mathrm{H}^1(G_T, U)$ and $\mathrm{H}^1(G_v, U)$ are affine algebraic varieties over $\mathbb{Q}_p$. Kim also shows that the localization maps are variety morphisms and that the crystalline torsors have the structure of (the $\mathbb{Q}_p$-points on) a subvariety.

In analogy with the classical theory of Selmer groups, we can cut down $\mathrm{H}^1(G_T, U)$ by local conditions to find a pointed set containing the images of the rational points, which we hope to be able to compute.

Definition 4.1 ([BD18, Definition 2.2])**.** We define the *Selmer variety* $\mathrm{Sel}(U)$ to be the reduced scheme associated to the subscheme of $\mathrm{H}^1(G_T, U)$ containing those classes c such that

- $\mathrm{loc}_p(c)$ is crystalline,
- $\mathrm{loc}_\ell(c) \in j_{U,\ell}(X(\mathbb{Q}_\ell))$ for all $\ell \neq p$,
- the projection of c to $\mathrm{H}^1(G_T, V)$ comes from an element of $J(\mathbb{Q}) \otimes \mathbb{Q}_p$.

See Kim [Kim09] for a proof that $\mathrm{H}^1_f(G_p, U)$ is a subvariety of $\mathrm{H}^1(G_p, U)$; it also follows from loc. cit. and [KT08] that $\mathrm{Sel}(U)$ is a subvariety of $\mathrm{H}^1(G_T, U)$ (see [Kim12] for an explanation why the conditions above might not produce a reduced scheme). The relation between this (slightly non-standard) definition of the Selmer variety and other definitions in the literature is discussed in [BD18, Remark 2.3]. Definition 4.1 has the convenient feature that our results will not depend on finiteness of the p-primary part of the Shafarevich-Tate group of $J/\mathbb{Q}$.

Since $j_{U,p}(X(\mathbb{Q}_p)) \subset \mathrm{H}^1_f(G_p, U)$ by Olsson's comparison theorem [Ols11, Theorem 1.4] in non-abelian p-adic Hodge theory, we obtain another commutative diagram

$$\begin{array}{ccc} X(\mathbb{Q}) & \longrightarrow & X(\mathbb{Q}_p) \\ \downarrow{\scriptstyle j_U} & & \downarrow{\scriptstyle j_{U,p}} \\ \mathrm{Sel}(U) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, U). \end{array} \tag{42}$$

Remark 4.2. Note that under our definitions, $\mathrm{Sel}(U)$ need not be contained in $\mathrm{H}^1_f(G_T, U)$, because $j_{U,\ell}(z)$ need not be unramified for all $\ell \neq p$ and $z \in X(\mathbb{Q}_\ell)$. In contrast to Bloch–Kato's foundational paper [BK90] and much of the subsequent literature, many papers in non-abelian Chabauty do not require unramified away from p in their definition of the global H^1_f. Following [BD21], we will try to avoid confusion by writing $\mathrm{H}^1_{f,S}(G_T, U)$ to mean the subvariety containing those classes that are crystalline at p and unramified at all primes ℓ that are not in S, where we will always choose S to be a subset of T_0. In this notation, we have $\mathrm{H}^1_{f,\varnothing}(G_T, U) = \mathrm{H}^1_f(G_T, U)$ and $\mathrm{Sel}(U) \subset \mathrm{H}^1_{f,T_0}(G_T, U)$ (but see Lemma 4.4 below).

Although the image of $j_{U,\ell}$ can be ramified, we have the following result.

Theorem 4.3 (Kim–Tamagawa [KT08])**.** *Suppose that $\ell \neq p$. Then the image $j_{U,\ell}(X(\mathbb{Q}_\ell))$ is finite. For a prime ℓ of good reduction for X, the image is trivial.*

This crucial result will enable us to control certain local heights away from p in a manner somewhat similar to Proposition 2.35. In fact we will use the following generalization of the second statement.

Lemma 4.4 ([BD18, Lemma 5.4])**.** *If X has potentially good reduction at ℓ, then $j_{U,\ell}(X(\mathbb{Q}_\ell))$ is trivial. Hence*

$$\mathrm{Sel}(U) \subset \mathrm{H}^1_{f,T_0'}(G_T, U),$$

where T_0' is the set of bad primes of X where X has potentially good reduction.

We define

$$X(\mathbb{Q}_p)_U := j_p^{-1}\left(\mathrm{loc}_p \mathrm{Sel}(U)\right) \subset X(\mathbb{Q}_p).$$

By commutativity of the diagram (42), we have that $X(\mathbb{Q}) \subset X(\mathbb{Q}_p)_U$.

Now suppose that U is a Galois-stable quotient of the maximal n-unipotent (i.e. having unipotency index $\leq n$) quotient of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}},b})_{\mathbb{Q}_p}$, which we denote by U_n. Then we obtain

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)_n := X(\mathbb{Q}_p)_{U_n} \subset X(\mathbb{Q}_p)_U\,. \tag{43}$$

Of course this is only useful for the purpose of computing rational points if $X(\mathbb{Q}_p)_U$ is finite and can be computed in practice. Kim conjectured that the first condition is eventually satisfied:

Conjecture 4.5 (Kim [Kim09])**.** *For $n \gg 0$, $X(\mathbb{Q}_p)_n$ is finite.*

There is very strong evidence for this conjecture, as shown in [Kim09, Section 3]. It is implied by a special case of the conjecture of Bloch–Kato (and other standard conjectures on motives).

For computational purposes this is already sufficient, once we can compute $X(\mathbb{Q}_p)_n$ as in Conjecture 4.5. The reason is that the Mordell–Weil sieve, discussed in §2.3.2 can often be used to show that given p-adic points do not come from a rational point. However, one of the most exciting potential applications of Kim's ideas would be an effective version of the Mordell conjecture. But while heuristics imply that the Mordell–Weil sieve should always work eventually [Poo06], it is not effective. The following stronger conjecture circumvents this issue.

Conjecture 4.6 (Kim [BDCKW18])**.** *For $n \gg 0$, $X(\mathbb{Q}_p)_n = X(\mathbb{Q})$.*

The n in Conjecture 4.6 may not be the n of Conjecture 4.5. They can differ already for the Chabauty–Coleman case $n = 1$. See recent work of Bianchi [Bia20] along these lines in the case of punctured elliptic curves.

Project 4.7 (Quadratic Chabauty and Kim's conjecture)**.** When $X/\mathbb{Q}$ is a genus g curve with $r = \mathrm{rk}\, J(\mathbb{Q}) = g - 1$, then typically the set of p-adic points $X(\mathbb{Q}_p)_1$ cut out by the Chabauty–Coleman method strictly contains $X(\mathbb{Q})$. In this project, we will first give an algorithm to compute the quadratic Chabauty set $X(\mathbb{Q}_p)_2$ under these hypotheses. Then we will investigate whether the quadratic Chabauty set, which satisfies

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)_2 \subset X(\mathbb{Q}_p)_1 \subset X(\mathbb{Q}_p),$$

is equal to $X(\mathbb{Q})$. (See [Bia20] for the case of integral points on punctured elliptic curves.) If $X(\mathbb{Q}) \neq X(\mathbb{Q}_p)_2$, we would like to characterize the points in $X(\mathbb{Q}_p)_2 \setminus X(\mathbb{Q})$. This project could be carried out on a database of genus 2 and 3 curves [The19].

Generalizing the étale formulation of classical Chabauty, Kim's approach is to show finiteness of $X(\mathbb{Q}_p)_U$ using p-adic Hodge theory. He obtains the following amendment of diagram (42):

$$\begin{array}{ccccc} X(\mathbb{Q}) & \longrightarrow & X(\mathbb{Q}_p) & & \\ \downarrow{\scriptstyle j_U} & & \downarrow{\scriptstyle j_{U,p}} & \searrow^{j_U^{\mathrm{dR}}} & \\ \mathrm{Sel}(U) & \xrightarrow{\mathrm{loc}_{U,p}} & \mathrm{H}^1_f(G_p, U) & \xrightarrow{\simeq} & U^{\mathrm{dR}}/\mathrm{Fil}^0 \end{array}$$

We refer to [Kim09] and Kim's lectures for the definitions of $U^{\mathrm{dR}} := D_{\mathrm{cris}}(U)$ (a quotient of Deligne's de Rham fundamental group $\pi_1^{\mathrm{dR}}(X_{\mathbb{Q}_p}, b)$), the isomorphism $\mathrm{H}^1_f(G_p, U) \to U^{\mathrm{dR}}/\mathrm{Fil}^0$ (coming from Olsson's comparison theorem [Ols11, Theorem 1.4]), and the locally analytic maps j_U^{dR} (these are iterated Coleman integrals). Note that this diagram specializes to Diagram (41) for $U = V$.

The analogue of the analytic properties of AJ_{b} (specifically that if there exists a nonzero functional we can construct that vanishes on $\overline{J(\mathbb{Q})}$ with Zariski dense image, given by a convergent p-adic power series then there are finitely many zeros on each residue disk of $X(\mathbb{Q}_p)$) is as follows:

Theorem 4.8 (Kim [Kim09])**.** *The map j_U^{dR} has Zariski dense image and is given by convergent p-adic power series on every residue disk.*

The analogue of the Chabauty–Coleman hypothesis $r < g$ is the non-density of $\mathrm{loc}_{U,p}$.

Theorem 4.9 (Kim [Kim09])**.** *Suppose that $\mathrm{loc}_{U,p}$ is non-dominant. Then $X(\mathbb{Q}_p)_U$ is finite.*

All known finiteness results come from bounding the dimension of $\mathrm{Sel}(U)$. For instance, Coates and Kim used Iwasawa theory to obtain dimension bounds in the following setting:

Theorem 4.10 (Coates–Kim [CK10])**.** *Let $X/\mathbb{Q}$ be a nice curve of genus $g \geq 2$ and suppose that J is isogenous over $\overline{\mathbb{Q}}$ to a product $\prod A_i$ of abelian varieties, with A_i having CM by a number field K_i of degree $2 \dim A_i$. Then $X(\mathbb{Q}_p)_n$ is finite for $n \gg 0$.*

Example 4.11. Theorem 4.10 can be used to show eventual finiteness in many nontrivial settings. For instance, Ellenberg and Hast use it to prove finiteness of rational points on solvable covers of $\mathbb{P}^1$ over $\mathbb{Q}$, see [EH22].

4.2. **Quadratic Chabauty.** Suppose that the set $X(\mathbb{Q}_p)_1$ cut out by classical Chabauty–Coleman is infinite. The goal of the quadratic Chabauty method is to

(a) show that $X(\mathbb{Q}_p)_2$ is finite;
(b) construct explicit functions on $X(\mathbb{Q}_p)$ cutting out (a finite set containing) $X(\mathbb{Q}_p)_2$.

In this section, we tackle (a), using Theorem 4.9. We will focus on (b) in Section 5.

Let $\rho(J)$ denote the *Picard number* of J, that is, the rank of $\mathrm{NS}(J)$, the Néron-Severi group of $J_{\mathbb{Q}}$. In this section, we will prove the following fundamental result.

Theorem 4.12 ([BD18, Lemma 3.2])**.** *Suppose that*

$$\mathrm{rk}\, J(\mathbb{Q}) < g + \rho(J) - 1\,. \tag{44}$$

Then $X(\mathbb{Q}_p)_2$ is finite.

We call (44) the *quadratic Chabauty condition*. The rough idea of the proof is to show that, assuming (44), there exists a Galois-stable quotient U of U_2 such that $\dim \mathrm{Sel}(U) < \dim \mathrm{H}^1_f(G_p, U)$. The result then follows from (43) and Theorem 4.9.

Remark 4.13. An alternative proof of Theorem 4.12 is due to Edixhoven–Lido [EL23], and is described in Lido's contribution to this volume. Their work uses the geometry of the Poincaré torsor of J, spread out over $\mathbb{Z}$. Yet another proof is due to Besser, Müller and Srinivasan [BMS21], based on a new construction of p-adic heights on line bundles on abelian varieties using p-adic Arakelov theory [Bes05]. It is somewhat closer to the quadratic Chabauty method for integral points presented in §2.3. Neither of these proofs uses Chabauty-Kim theory or p-adic Hodge theory.

In fact, we will first prove a simpler, but important special case.

Proposition 4.14. *Suppose that X has potentially good reduction everywhere. If $\operatorname{rk} J(\mathbb{Q}) = g$ and $\rho(J) > 1$, then $X(\mathbb{Q}_p)_2$ is finite.*

For instance, this suffices to prove finiteness of $X(\mathbb{Q}_p)_2$ for the split (or non-split) Cartan modular curve at level 13 [BDM+19].

4.3. **Dimension counts.** As mentioned above, finiteness proofs in Chabauty–Kim theory usually require bounds on the dimensions of Galois-cohomology groups. More precisely, for a quotient U as in §4.1, we need to compare the local dimension $\dim \mathrm{H}^1_f(G_p, U)$ with the global dimension $\dim \mathrm{Sel}(U)$. From now on, suppose that U is a Galois-stable quotient of U_2 which sits in a Galois-equivariant short exact sequence

$$1 \to [U,U] \to U \to V \to 1\,, \tag{45}$$

where $V = \mathrm{H}^1_{\text{ét}}(\bar{X}, \mathbb{Q}_p)$. The reason for this choice will become apparent soon. Of course we don't want to take $U = V$, since that would only recover Chabauty's result. The idea is to choose U to be "slightly non-abelian". In order to do so, we first compute $\dim \mathrm{H}^1_f(G_p, U)$ and bound $\dim \mathrm{Sel}(U)$ in terms of data depending only on $[U,U]$; this will then suggest to pick a quotient U such that $[U,U] \simeq \mathbb{Q}_p(1)$ (or a direct sum thereof).

We start with the local computation.

Lemma 4.15. *We have*

$$\dim \mathrm{H}^1_f(G_p, U) = \dim \mathrm{H}^1_f(G_p, [U,U]) + g.$$

Proof. All representations in (45) are de Rham, so we obtain a short exact sequence

$$1 \to D_{\mathrm{dR}}([U,U])/F^0 \to D_{\mathrm{dR}}(U)/F^0 \to D_{\mathrm{dR}}(V)/F^0 \to 1\,. \tag{46}$$

Since $\phi = \mathrm{id}$ for $W \in \{[U,U], V, U\}$, we have the Bloch–Kato logarithm

$$\mathrm{H}^1_f(G_p, W) \simeq D_{\mathrm{dR}}(W)/F^0$$

from Remark 3.12. By [Kim05, Section 1], this is an algebraic isomorphism of schemes. Therefore we deduce

$$\dim \mathrm{H}^1_f(G_p, U) = \dim \mathrm{H}^1_f(G_p, [U,U]) + \dim \mathrm{H}^1_f(G_p, V) \tag{47}$$

from (46). But $\mathrm{H}^1_f(G_p, V) \simeq \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)$, so the result follows. □

We now turn to the dimension of $\mathrm{Sel}(U)$. We have that $\mathrm{H}^0(G_T, W) = 0$, for all terms in (45), so the corresponding six-term exact sequence of non-abelian Galois cohomology (see Proposition A.2) induces an exact sequence

$$\mathrm{H}^1(G_T, [U,U]) \to \mathrm{H}^1(G_T, U) \to \mathrm{H}^1(G_T, V)\,.$$

It is shown in [Kim05] that this is an exact sequence of pointed varieties, inducing another exact sequence

$$\mathrm{H}^1_f(G_T, [U,U]) \to \mathrm{H}^1_f(G_T, U) \to \mathrm{H}^1_f(G_T, V) \tag{48}$$

of pointed varieties (note that this is in general weaker than the local version leading to (47)).

For now let's assume that X has potentially good reduction everywhere. This implies that any class $c \in \mathrm{Sel}(U)$ satisfies $\mathrm{loc}_\ell(c) = 0$ (and, in particular, is unramified) for all $\ell \neq p$; hence

$$\mathrm{Sel}(U) \subset \mathrm{H}^1_{f,\varnothing}(G_T, U) = \mathrm{H}^1_f(G_T, U). \tag{49}$$

This is where we use the third requirement in Definition 4.1: we may now conclude

$$\dim \mathrm{Sel}(U) \leq \mathrm{rk}\, J(\mathbb{Q}) + \dim \mathrm{H}^1_f(G_T, [U,U]) \tag{50}$$

from (48) and (49) without any finiteness assumptions on the Shafarevich-Tate group.

To prove non-density of the localization map, we want $\mathrm{H}^1_{f,\varnothing}(G_T, U)$ (or $\mathrm{H}^1_{f,T_0'}(G_T, U)$ if we don't have potentially good reduction) to be small, and $\mathrm{H}^1_f(G_p, U)$ to be large. Hence it is natural to look for quotients U such that $[U,U] \simeq \mathbb{Q}_p(1)$, since in this case Example 3.3, Example 3.4, Lemma 4.15 and (50) imply:

Lemma 4.16. *Suppose that X has potentially good reduction everywhere. If $\mathrm{rk}\, J(\mathbb{Q}) \leq g$ and $[U,U] \simeq \mathbb{Q}_p(1)$, then $X(\mathbb{Q}_p)_U$ is finite.*

We will generalize Lemma 4.16 below. For now, let us note:

Corollary 4.17. *Suppose that X has potentially good reduction everywhere and that $\mathrm{rk}\, J(\mathbb{Q}) \leq g$. If there exists a Galois stable quotient U of U_2 such that $[U,U] \simeq \mathbb{Q}_p(1)$, then $X(\mathbb{Q}_p)_2$ is finite.*

4.4. **Constructing a $\mathbb{Q}_p(1)$-quotient of U_2.** We will now show that a quotient of U_2 as in Corollary 4.17 exists when the Picard number of J is strictly greater than 1.

Lemma 4.18. *Suppose that $\rho(J) > 1$. Then there exists a Galois-stable quotient U of U_2 surjecting onto V such that $[U,U] = \mathbb{Q}_p(1)$.*

Combining Corollary 4.17 and Lemma 4.18, we deduce Proposition 4.14.

All known methods to construct such a quotient U use a geometric approach. We will follow [BDM+19] in phrasing the construction in terms of a correspondence $Z \subset X \times X$. See [Smi05, Chapter 3] for background on correspondences and [Mil80, Chapter VI.9] for background on cycle classes. Applying the Künneth projector to the cycle class in $H^2(\bar{X} \times \bar{X})$ of Z, we obtain

$$\xi_Z \in \mathrm{H}^1(\bar{X}, \mathbb{Q}_p) \otimes \mathrm{H}^1(\bar{X}, \mathbb{Q}_p)(1) \simeq \mathrm{End}\, \mathrm{H}^1(\bar{X}, \mathbb{Q}_p).$$

Definition 4.19. A nontrivial correspondence $Z \subset X \times X$ is *nice* if

(i) there are $c_1, c_2 \in \mathrm{Pic}(X)$ such that the pushforward of the class $[Z] \in \mathrm{Pic}(X \times X)$ under the canonical involution $(x,y) \mapsto (y,x)$ is $[Z] + \pi_1^* c_1 + \pi_2^* c_2$ where π_1, π_2 are the canonical projections,
(ii) the class ξ_Z, viewed as an endomorphism of $\mathrm{H}^1(\bar{X}, \mathbb{Q}_p)$, has trace zero.

Note that a correspondence satisfying (i) induces an endomorphism of J fixed by the Rosati involution, i.e. a nontrivial element of $\mathrm{NS}(J)$. This proves the first part of the following result. The second part follows from (ii), since the trace factors through the cup product.

Lemma 4.20 ([BDM+19, Lemma 2.4])**.** *Suppose J is absolutely simple, and let $Z \subset X \times X$ be a correspondence satisfying (i) above. Then ξ_Z lies in the subspace*

$$\bigwedge^2 \mathrm{H}^1(\bar{X}, \mathbb{Q}_p)(1) \subseteq \mathrm{H}^1(\bar{X}, \mathbb{Q}_p) \otimes \mathrm{H}^1(\bar{X}, \mathbb{Q}_p)(1).$$

Moreover Z is nice if and only if the image of ξ_Z in $H^2(\bar{X}, \mathbb{Q}_p)(1)$ under the cup product is zero.

Composing the cycle class map with the Künneth projector, we therefore get a morphism

$$c_Z\colon \mathbb{Q}_p(-1) \to \ker\left(\bigwedge^2 \mathrm{H}^1(\bar{X},\mathbb{Q}_p) \xrightarrow{\cup} H^2(\bar{X},\mathbb{Q}_p)\right) \tag{51}$$

for every nice correspondence Z.

Proof of Lemma 4.18. Let $U[2] := \ker(U_2 \to U_1 = V)$, so that we have an exact sequence

$$1 \to U[2] \to U_2 \to V \to 1.$$

A Galois-stable quotient of U_2 surjecting onto V is therefore of the form U_2/W, where W is a Galois-stable subrepresentation of $U[2]$. So if we have a Galois-equivariant morphism $\gamma\colon U[2] \to \mathbb{Q}_p(1)$, then we can form a suitable quotient $U := U_2/\ker\gamma$ of U_2 via pushout:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U[2] & \longrightarrow & U_2 & \longrightarrow & V & \longrightarrow & 1 \\ & & {\scriptstyle\gamma}\downarrow & & \downarrow & & {\scriptstyle=}\downarrow & & \\ 1 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 1 \end{array}$$

To describe the representation $U[2]$, note that there is an anti-symmetric pairing

$$V \times V = U_1 \times U_1 \to U[2]$$

induced by the commutator map. It is surjective with kernel equal to the image of $H^2_{\text{ét}}(\bar{X},\mathbb{Q}_p)$ inside $\wedge^2 V$ under the dual of the cup product $\cup\colon \wedge^2 \mathrm{H}^1(\bar{X},\mathbb{Q}_p) \to H^2_{\text{ét}}(\bar{X},\mathbb{Q}_p)^*$, so the Galois representation $U[2]$ can be described via an exact sequence

$$1 \to H^2_{\text{ét}}(\bar{X},\mathbb{Q}_p) \xrightarrow{\cup^*} \wedge^2 V \to U[2] \to 1.$$

Hence we want a morphism

$$\gamma\colon \operatorname{Coker}(\cup^*\colon H^2_{\text{ét}}(\bar{X},\mathbb{Q}_p) \to \wedge^2 V) \to \mathbb{Q}_p(1).$$

By Lemma 4.20, if Z is a nice correspondence, then we can take $\gamma := c_Z^*(1)$, where c_Z^* is the dual of the map in (51). Then $U := U_2/\ker c_Z^*(1)$ has the desired properties. Lemma 4.20 also implies that the subspace of $\mathrm{Pic}(X \times X) \otimes \mathbb{Q}$ consisting of 0 and the classes of nice correspondences has dimension $\rho(J) - 1$, completing the proof. □

In the following, we denote the quotient $U_2/\ker c_Z^*(1)$ associated to a nice correspondence Z by U_Z.

Remark 4.21. By the above, we can think of U as coming from a nontrivial cycle $Z \in \ker(\widetilde{\mathrm{AJ}}^*)$, where

$$\widetilde{\mathrm{AJ}}^*\colon \mathrm{NS}(J) \to \mathrm{NS}(X \times X) \to \mathrm{NS}(X)$$

is as in [DLF21, Section 2].

Remark 4.22. Another construction of U_Z is described in [BBB+21, §4.2]. It closely resembles the approach of Edixhoven–Lido [EL23]. Roughly speaking, we start with a cycle $Z \in \ker \widetilde{\mathrm{AJ}}^*$ as above, lift it to a line bundle L_Z on J whose restriction to X is trivial and we let U_Z denote the $\mathbb{Q}_p$-étale fundamental group of the $\mathbb{G}_m$-torsor L_Z^*. This yields the same U_Z as the one in the proof above, and probably (after taking a multiple and extending to the Néron model) the same $\mathbb{G}_m$-torsor $\mathcal{L}_Z$ as in Edixhoven-Lido. Betts [Beta] constructs local p-adic heights on $\mathcal{L}_Z(\mathbb{Q}_p)$, factoring through $\mathrm{H}^1_f(G_p, U_Z)$; his results could be used as an alternative way to our approach for computing local p-adic heights on $\mathrm{H}^1_f(G_p, U_Z)$ discussed below.

4.5. **Beyond potentially good reduction: the twisting construction.** To finish the proof of Theorem 4.12, we need to generalize Lemma 4.16 by

- allowing $[U,U] \simeq \mathbb{Q}_p(1)^{\oplus n}$, where $0 < n < \rho(J)$, rather than requiring $[U,U] \simeq \mathbb{Q}_p(1)$;
- removing the condition that X has potentially good reduction everywhere.

The first of these is trivial, since we have

$$\dim \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)^{\oplus n}) = n$$

by Example 3.3 and

$$\dim \mathrm{H}^1_f(G_T, \mathbb{Q}_p(1)^{\oplus n}) = 0$$

by Example 3.4. Moreover, the proof of Lemma 4.18 can be amended easily to show that we can always construct a suitable quotient U such that $[U,U] \simeq \mathbb{Q}_p(1)^{\oplus \rho(J)-1}$. Hence the proof of Theorem 4.12 is complete once we show the following result:

Lemma 4.23. *Let U be a Galois-stable quotient of U_2 which sits in a Galois-equivariant short exact sequence (45). Then we have*

$$\dim \mathrm{Sel}(U) \leq \mathrm{rk}\, J(\mathbb{Q}) + \dim \mathrm{H}^1_f(G_T, [U,U]). \tag{52}$$

Proof. Note that (52) is a generalization of (50). Recall that to prove (50), we used that $\mathrm{Sel}(U) \subset \mathrm{H}^1_f(G_T, U)$, which might not hold in general, see Remark 4.2. To remedy this, suppose that $\alpha = (\alpha_\ell)_\ell \in \prod_{\ell \in T_0} j_\ell(X(\mathbb{Q}_\ell))$ is a set of *local conditions* such that $\alpha_\ell \in j_\ell(X(\mathbb{Q}_\ell))$ is ramified for some ℓ. Let $\mathrm{Sel}(U)_\alpha$ denote the preimage of α under $\prod_{\ell \in T_0} \mathrm{loc}_\ell$ and let $\beta \in \mathrm{Sel}(U)_\alpha$. The idea is to to use the twisting construction in non-abelian cohomology (see Appendix A) to show that $\mathrm{Sel}(U)_\alpha$ is isomorphic to a subvariety $\mathrm{H}^1_f(G_T, U^{(\beta)})'$ of $\mathrm{H}^1_f(G_T, U^{(\beta)})$. There is an analogue of the exact sequence (48) for $U^{(\beta)}$, leading to an upper bound

$$\dim \mathrm{H}^1_f(G_T, U^{(\beta)})' \leq \dim \mathrm{H}^1_f(G_T, U^{(\beta)}) \leq \dim \mathrm{H}^1_f(G_T, V) + \dim \mathrm{H}^1_f(G_T, [U,U]) \tag{53}$$

and hence

$$\dim \mathrm{Sel}(U)_\alpha \leq \mathrm{rk}\, J(\mathbb{Q}) + \dim \mathrm{H}^1_f(G_T, [U,U]).$$

Since $\mathrm{Sel}(U)$ is the disjoint union of finitely many $\mathrm{Sel}(U)_\alpha$ by Theorem 4.3, this proves the lemma.

We give a bit more detail. Letting U act on itself by conjugation, we form the twist $U^{(\beta)}$ of U by the U-torsor β. Let

$$h\colon \mathrm{H}^1(G_T, U) \to \mathrm{H}^1(G_T, U^{(\beta)})$$

denote the bijection from Proposition A.3, sending β to the trivial class. Then h maps crystalline classes to crystalline classes and preimages of $J(\mathbb{Q}) \otimes \mathbb{Q}_p$ to preimages of $J(\mathbb{Q}) \otimes \mathbb{Q}_p$, see the proof of [BD18, Lemma 2.6]. We define $\mathrm{H}^1_f(G_T, U^{(\beta)})'$ to be the reduced subscheme of $\mathrm{H}^1(G_T, U^{(\beta)})$ representing classes c such that

- $\mathrm{loc}_p(c)$ is crystalline
- $\mathrm{loc}_\ell(c) = 0$ for all $\ell \neq p$
- the projection of c to $\mathrm{H}^1(G_T, V)$ comes from an element of $J(\mathbb{Q}) \otimes \mathbb{Q}_p$

(compare with Definition 4.1). By the first two items, we have $\mathrm{H}^1_f(G_T, U^{(\beta)})' \subset \mathrm{H}^1_f(G_T, U^{(\beta)})$, and the discussion above shows that $h(\mathrm{Sel}(U)_\alpha) \subset \mathrm{H}^1_f(G_T, U^{(\beta)})'$.

Now consider the twists $[U,U]^{(\beta)}$ and $V^{(\beta)}$, where, as above, U acts by conjugation. Since this action is unipotent, the two twisting morphisms are Galois-equivariant group isomorphisms. Hence the twisting construction turns (48) into a Galois-equivariant exact sequence

$$1 \to [U,U] \to U^{(\beta)} \to V \to 1,$$

resulting, via Kim's arguments in [Kim05] as in the discussion following Lemma 4.15, in an exact sequence of pointed varieties

$$\mathrm{H}^1(G_T,[U,U]) \to \mathrm{H}^1(G_T,U^{(\beta)}) \to \mathrm{H}^1(G_T,V)$$

and, via [Kim09], in another exact sequence of pointed varieties

$$\mathrm{H}^1_f(G_T,[U,U]) \to \mathrm{H}^1_f(G_T,U^{(\beta)}) \to \mathrm{H}^1_f(G_T,V)\,. \tag{54}$$

In the above, we are using that by Remark A.4 the twisting morphism h is an isomorphism of schemes, since $\mathrm{H}^1(G_T,W)$ and $\mathrm{H}^1(G_T,W^{(\beta)})$ are affine schemes for $W \in \{[U,U],U,V\}$ by [Kim09]. Finally, (53) follows from (54) just like (50). □

4.6. **Extending the quadratic Chabauty Lemma.** Theorem 4.12 has been extended in several ways. First, there is an obvious extension to curves over imaginary quadratic fields, and in fact [BD18, Lemma 3.2] already includes this case. One needs to restrict to such fields because of the crucial use of Example 3.4.

In [DLF21], Dogra and Le Fourn extend Theorem 4.12 to Jacobians admitting an isogeny $J \to A \times B$ defined over $\mathbb{Q}$ such that $\mathrm{Hom}(A,B) = 0$ and satisfying a condition similar to the quadratic Chabauty condition, but phrased in terms of A and B. See [DLF21, Proposition 1.6] for the precise statement. They use this result to show that $X(\mathbb{Q}_p)_2$ is finite for the nonsplit Cartan modular curve at prime level $N \neq p$, whenever this curve has genus at least 2 and at least one rational point.

In [BD21], Balakrishnan and Dogra weaken the rank condition by replacing $\rho(J)$ by $\rho(J) + \operatorname{rk}\mathrm{NS}(J^{c=-1}_{\bar{\mathbb{Q}}})$, where c denotes complex conjugation. In this setting, one needs to allow more general $[U,U]$. This essentially exhausts the Artin-Tate part of $[U_2,U_2]$, so different ideas are needed to prove finiteness of $X(\mathbb{Q}_p)_2$ for more general curves.

Balakrishnan and Dogra show in [BD21, Lemma 2.6] that a special case of the Bloch–Kato conjecture [BK90, Conjecture 5.3(i)], applied to $X \times X$, implies that $X(\mathbb{Q}_p)_2$ is finite for $\operatorname{rk} J(\mathbb{Q}) < g^2$, independently of $\mathrm{NS}(J_{\mathbb{Q}})$. In the proof, they work directly with U_2, rather than a quotient thereof.

5. Computing with quadratic Chabauty

Let $X/\mathbb{Q}$ be a nice curve of genus $g \geq 2$. In the previous section, we showed that when X satisfies the quadratic Chabauty condition (44), then there is a Galois-stable quotient $U = U_Z$ of U_2, depending on a nice correspondence $Z \subset X \times X$, such that $X(\mathbb{Q}_p)_U$ is finite (and hence $X(\mathbb{Q}_p)_2$ is finite as well). We now discuss how to compute $X(\mathbb{Q}_p)_U$ in practice.

Recall the situation discussed in §2.3, in particular Corollary[15] 2.36: When $X\colon y^2 = f(x)$ is hyperelliptic and satisfies some additional conditions, then for $P \in X(\mathbb{Q})$, the Coleman–Gross p-adic height pairing satisfies

$$h((P)-(\infty),(P)-(\infty)) = h_p((P)-(\infty),(P)-(\infty)) + \sum_{\ell \neq p} h_\ell((P)-(\infty),(P)-(\infty))\,.$$

We showed that

(i) the function $P \mapsto h_p((P)-(\infty),(P)-(\infty))$ extends to a function $\theta\colon X(\mathbb{Q}_p)\setminus\{\infty\} \to \mathbb{Q}_p$, locally analytic away from the disk at infinity;
(ii) $h_\ell((z)-(\infty),(z)-(\infty)) = 0$ for *integral*[16] points $z \in X(\mathbb{Q}_\ell)$;

[15] More generally, see Theorem 2.34.
[16] See Proposition 2.35 for a more general statement.

(iii) h is a symmetric bilinear pairing on $J(\mathbb{Q}) \otimes \mathbb{Q}_p$, and hence can be written as a linear combination of a basis of such pairings.

More precisely, the local height h_p had an interpretation as a sum of double Coleman integrals, which can be thought of as a solution to a p-adic differential equation. Since we assumed in Corollary 2.36 that log restricts to an isomorphism $J(\mathbb{Q}) \otimes \mathbb{Q}_p \to \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$, we can construct a basis in (iii) via products of single integrals. These are restrictions of locally analytic functions on $J(\mathbb{Q}_p)$, so by pullback we get a function $\rho\colon X(\mathbb{Q}_p) \to \mathbb{Q}_p$ from (i) and (iii) with finitely many zeros among the local integral points which vanishes in the global integral points by (ii).

Remark 5.1. The presence of double Coleman integrals suggests that we have found some part of $X(\mathbb{Z}_p)_2$, the "quadratic" or depth 2 part of Kim's nonabelian Chabauty, since $X(\mathbb{Q}_p)_n$ or $X(\mathbb{Z}_p)_n$ are cut out by n-fold iterated integrals [Kim09, BDCKW18]. This was proved by Balakrishnan and Dogra, see [BD18, Remark 3.3, Remark 6.4, Lemma 7.6].

One difficulty in extending this construction to *rational points* is that we do not have a good way to control $\sum_{\ell \neq p} h_\ell((P) - (\infty), (P) - (\infty))$ (or a similar local height in the non-hyperelliptic case) for general $P \in X(\mathbb{Q})$. This is where we use Chabauty–Kim. As in Section 4, we assume that the quadratic Chabauty condition (44) is satisfied, and we fix a Galois-stable quotient $U = U_Z$ of U_2, where $Z \subset X \times X$ is a nice correspondence. Recall that for $\ell \neq p$, the image $j_{U,\ell}(X(\mathbb{Q}_\ell))$ inside $\mathrm{H}^1(G_\ell, U)$ is finite by Theorem 4.3, and is trivial if X has potentially good reduction at ℓ by Lemma 4.4. Therefore our goal is to construct (local and global) p-adic heights which factor through Kim's unipotent Kummer maps $j_{U,v}$ and such that the local height function at p is locally analytic along all of $X(\mathbb{Q}_p)$. It turns out that Nekovář's construction of p-adic heights in terms of p-adic Hodge theory, as discussed in Section 3, makes this possible, via the twisting construction in non-abelian cohomology.

As in Section 4, we denote

$$V = \mathrm{H}^1_{\text{ét}}(\bar{X}, \mathbb{Q}_p)\,.$$

Recall that for a prime v the local Nekovář-height h_v is defined on mixed extensions of p-adic G_v-representations with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$. We will discuss in §5.1 that we can take a torsor $P \in \mathrm{H}^1(G_v, U)$ and create such a mixed extension $\tau_v(P)$ (depending on Z). In fact, the same construction produces a global mixed extension $\tau(P)$ for $P \in \mathrm{H}^1(G_T, U)$ (recall that G_T is the maximal quotient of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside $T = T_0 \cup \{p\}$, where T_0 is the set of bad primes for X). This implies that the function

$$\mathrm{Sel}(U) \to \mathbb{Q}_p\,; \quad P \mapsto h_\ell \circ \tau_\ell \circ \mathrm{loc}_\ell(P) \tag{55}$$

has finite image for $\ell \neq p$, and if X has potentially good reduction at ℓ, then the image of the map (55) is trivial.

To ease notation, we write

$$A(x) := \tau_v(j_{U,v}(x))$$

for $x \in X(\mathbb{Q}_v)$. This assigns a mixed extension of G_v-representations with graded pieces $\mathbb{Q}_p, V$ and $\mathbb{Q}_p(1)$ to a $\mathbb{Q}_v$-rational point on X. Similarly, we write $A(x) := \tau(j_U(x))$ for $x \in X(\mathbb{Q})$.

We now sketch how quadratic Chabauty works to study rational points on curves. More technical details and computational issues are discussed further below. We use this to give an analogue of Theorem 2.34 in the simplest situation covered by Theorem 4.12, namely

(i) $r = g > 1$,

(ii) $\log\colon J(\mathbb{Q}) \otimes \mathbb{Q}_p \to \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ is an isomorphism[17], and
(iii) $\operatorname{rk} \mathrm{NS}(J_{\mathbb{Q}}) > 1$.

By (i), the Kummer map $J(\mathbb{Q}) \otimes \mathbb{Q}_p \to \mathrm{H}^1_f(G_{\mathbb{Q}}, V)$ in Diagram (41) is an isomorphism. Since the Bloch-Kato logarithm $\log_{\mathrm{BK}}\colon \mathrm{H}^1_f(G_p, V) \to \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ is also an isomorphism, (ii) implies that

$$\mathrm{H}^1_f(G_{\mathbb{Q}}, V) \xrightarrow{\mathrm{loc}_p} \mathrm{H}^1_f(G_p, V) \xrightarrow{\log_{\mathrm{BK}}} \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$$

is an isomorphism, so by Poincaré duality we may view the global height h as a bilinear pairing

$$h\colon \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^* \times \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^* \to \mathbb{Q}_p\ .$$

By abuse of notation, we may replace the target of the projection maps π_1, π_2 introduced in (25) by $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$.

As in Section 3.2, we fix

(a) a continuous nontrivial idèle class character $\chi\colon \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^* \to \mathbb{Q}_p$,
(b) a splitting s of the Hodge filtration on V_{dR} such that $\ker(s)$ is isotropic with respect to the dual of the cup product pairing.

Recall from Remark 3.6 that the isotropicity condition implies that h is symmetric. The following result sums up the theoretical foundation for our approach to computing rational points via p-adic heights:

Theorem 5.2 (Quadratic Chabauty for rational points [BD18, Proposition 5.5])**.** *Let $X/\mathbb{Q}$ be a nice curve of genus $g > 1$, let J be the Jacobian of X, assume that $\operatorname{rk} J(\mathbb{Q}) = g$, and that $\rho(J) > 1$. Choose a prime p of good reduction for X such that $\log\colon J(\mathbb{Q}) \otimes \mathbb{Q}_p \to \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ is an isomorphism. Choose a nice correspondence Z on X and let $U = U_Z$. Finally, fix a basis $\psi_1, \dots, \psi_N$ of the space*

$$(\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^* \otimes \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*)^*$$

of $\mathbb{Q}_p$-valued bilinear forms on $(\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$. Then there exist constants $\alpha_i \in \mathbb{Q}_p$ such that the function $X(\mathbb{Q}_p) \to \mathbb{Q}_p$ defined by

$$\rho(x) := h_p(A(x)) - \sum_{i=1}^{N} \alpha_i \psi_i \circ (\pi_1, \pi_2)(A(x)) \tag{56}$$

has finitely many zeros and takes values in a finite set $\mathcal{S} \subset \mathbb{Q}_p$ on $X(\mathbb{Q}_p)_U$.

If X has everywhere potentially good reduction, then this holds for $S = \{0\}$.

Proof. As in the proof of Theorem 2.34, we can find constants $\alpha_i \in \mathbb{Q}_p$ such that

$$h = \sum_i \alpha_i \psi_i\ .$$

For $\ell \neq p$ let $\mathcal{S}_\ell$ be the image of the local height in (55). By the discussion above, the sets $\mathcal{S}_\ell$ are finite, and we have $\mathcal{S}_\ell = \varnothing$ if X has potentially good reduction at ℓ. Then, if we define ρ by (56), we have that $\rho(X(\mathbb{Q}_p)_U)$ is contained in the finite set $\mathcal{S} := \sum_{\ell \neq p} \mathcal{S}_\ell \subset \mathbb{Q}_p$ by construction.

To show that ρ only has finitely many zeros, it suffices to prove that the map

$$(\mathrm{AJ}_{\mathrm{b}}, h_p \circ A)\colon X(\mathbb{Q}_p) \to \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^* \times \mathbb{Q}_p \tag{57}$$

has Zariski dense image. Recall that A factors as $\tau_p \circ j_{U,p}$; we also factor $\mathrm{AJ}_{\mathrm{b}} = \pi_* \circ j_{U,p}$. By [BD21, Lemma 3.10],

$$(\pi_*,\ h_p \circ \tau_p)\colon \mathrm{H}^1_f(G_p, U) \longrightarrow \mathrm{H}^1_f(G_p, V) \times \mathbb{Q}_p$$

[17] If this fails, we can simply use Chabauty–Coleman to compute the rational points.

is an isomorphism of schemes. Since $j_{U,p}$ has Zariski dense image (see [Kim09] and Kim's lectures), the result follows.

□

Remark 5.3. In fact we have the following exact equality by [BD21, Lemma 4.1]:

$$X(\mathbb{Q}_p)_U = \{x \in X(\mathbb{Q}_p) \,:\, \rho(x) \in \mathcal{S}\}\,.$$

Note the similarities with Theorem 2.34.

To fill in the gaps in the discussion above, we will construct τ (and τ_p) in the next subsection. This will be done by first constructing a mixed extension A_Z as a quotient of the universal enveloping algebra of the Lie algebra of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}})_{\mathbb{Q}_p}$, following the construction of U_Z in §4.4. Then we define τ via the twisting construction in non-abelian cohomology.

In §5.2, we discuss how to compute $\pi_1(A(x))$, $\pi_2(A(x))$ and $h_p(A(x))$ for $x \in X(\mathbb{Q}_p)$. Further computational issues are addressed in §5.3; there, we also give an algorithmic version of Theorem 5.2 (Algorithm 5.27) and how the coefficients α_i can be derived for a suitable basis $\{\psi_i\}$. In particular, this will show:

Corollary 5.4. *In the situation of Theorem 5.2, the function ρ is explicitly computable.*

We explain some features of our `Magma`-code for quadratic Chabauty (for modular curves) in §5.4. Finally, we give a worked example, determining rational points on the Atkin-Lehner quotient $X_0(167)/w_{67}$ in §5.5, and we discuss some subsequent computational work on the quadratic Chabauty method, including the results of two projects proposed in 2020 in §5.6.

Remark 5.5. In Theorem 5.2, the dependence on Z, s and χ is hidden by our notation. In fact, both the mixed extensions $A(x)$ (and thus the function ρ) and the set $\mathcal{S}$ depend on Z. Moreover, ρ depends on the choices of χ and s, and $\mathcal{S}$ depends on χ.

Remark 5.6. Theorem 5.2 does not describe how the set $\mathcal{S}$ can be computed. In general, this is a difficult problem, but in certain cases of interest, we can show that $\mathcal{S}$ is trivial. See §5.3.1.

Remark 5.7. Nekovář attaches in [Nek93, §5] a mixed extension of Galois representation to a pair of divisors of degree 0 on the curve with disjoint support. In [BD18, Section 6], Balakrishnan and Dogra show that for v prime and $x \in X(\mathbb{Q}_v)$, the mixed extension $A(x)$ is in fact isomorphic to the mixed extension associated to the divisor $(x) - (b)$ and the divisor

$$D_Z(b, x) = \Delta^*(Z) - i_{1,b}^*(Z) - i_{2,x}^*(Z) \in \mathrm{Div}^0(X)\,,$$

where $i_{1,b}(y) = (y, b) \in X \times X$, $i_{2,x}(y) = (x, y)$ and $\Delta\colon X \to X \times X$ is the diagonal embedding. For this construction, one needs that Z does not intersect $(\Delta - X \times x_1 - x_2 \times X)$ for any pair $(x_1, x_2) \in X \times X$; a correspondence $Z \subset X \times X$ satisfies this condition if and only if it is nice. In order to make this approach explicit, one needs explicit formulas for Z, for instance as a divisor on $X \times X$. When X is a bielliptic curve of genus 2, then Z is a sum of sections, and the heights can be computed on the corresponding elliptic curves. They use this simpler concrete description to compute the rational points on a bielliptic genus 2 curve over $\mathbb{Q}$ and the $\mathbb{Q}(i)$-rational points on the bielliptic genus 2 curve $X_0(37)$, answering a question of Daniels and Lozano-Robledo. In general, one can obtain formulas for the correspondence Z using an algorithm due to Costa–Mascot–Sijsling–Voight [CMSV19]; see also recent work of Duque-Rosero, Hashimoto and Spelier [DRHS22], where equations for divisors $Z \subset X \times X$ obtained using [CMSV19] are used for geometric quadratic Chabauty.

Remark 5.8. A different proof of Theorem 5.2 is due to Besser, Müller and Srinivasan [BMS21]. They first show an analogous result for the p-adic height on the line bundle on J associated to Z, pulled back to a p-adic height on the trivial bundle on X. They then show that this recovers Theorem 5.2 by relating their local heights to $h_v((x)-(b), D_Z(b,x))$.

Remark 5.9. Let Z' be another nice correspondence. Then $X(\mathbb{Q}_p)_{U_Z}$ and $X(\mathbb{Q}_p)_{U_{Z'}}$ are equal if the classes of Z and Z' are dependent in $\ker(\mathrm{NS}(J) \to \mathrm{NS}(X))$, see [BD18, Remark 5.7]. In contrast, if the classes of Z and Z' are independent, then we expect that $X(\mathbb{Q}_p)_{U_Z} \cap X(\mathbb{Q}_p)_{U_{Z'}} = X(\mathbb{Q})$ unless there is a geometric reason for the intersection to be larger.

Remark 5.10. In the remainder of these notes, we discuss how Theorem 5.2 can be turned into a method for *computing* the rational points for a given curve X. However, it is also possible to *bound* $\#X(\mathbb{Q})$ for curves X satisfying our assumptions; see [BD19a]. See also [DLF21] for an extension. By endowing the algebra of Coleman functions on $X_{\mathbb{Q}_p}$ with a weight filtration, Betts has recently shown how to obtain effective Chabauty-Kim results from the computation of dimensions of certain Bloch-Kato Selmer groups (see [Betb]). In particular, this recovers and improves the bounds obtained by Balakrishnan and Dogra in [BD19a], but also produces far more general results. It was applied by Betts, Corwin and Leonhardt in [BC22] to obtain a general upper bound on $\#X(\mathbb{Q})$ for a nice curve $X/\mathbb{Q}$ of genus $g > 1$, conditional on finiteness of the Shafarevich-Tate group of $\mathrm{Jac}(X)/\mathbb{Q}$ and on the Bloch-Kato conjectures. They also give a more explicit bound for S-integral points on punctured elliptic curves with CM.

5.1. **Twisting and mixed extensions.** Let $X/\mathbb{Q}$ be a nice curve of genus $g > 1$ such that $\operatorname{rk}\mathrm{NS}(J_{\mathbb{Q}}) > 1$ and $X(\mathbb{Q}) \neq \varnothing$. Recall that for every finite prime v and every point $x \in X(\mathbb{Q}_v)$, we want to construct a mixed extension $A(x)$ of Galois representations with graded pieces $\mathbb{Q}_p$, V and $\mathbb{Q}_p(1)$ such that the local height function $x \mapsto h_v(A(x))$ factors through Kim's unipotent Kummer map $j_{U,v}$, where $U = U_Z$ is the fundamental group quotient corresponding to a nice correspondence Z, which we fix. We also fix a base point $b \in X(\mathbb{Q})$. We will first construct a mixed extension $A_Z = A_Z(b)$; then $A(b) = A_Z$, and for other points x, the mixed extension $A(x)$ will be obtained from A_Z using the twisting construction in nonabelian cohomology (in fact $A_Z = A(b)$). The construction of A_Z will be analogous to the construction of $U := U_Z$ in §4.4, but on the Lie algebra side. For more details, see [BD18, Section 5], and see [BD21, Sections 3,4] for a generalization.

Recall that U is a Galois-stable quotient is the maximal 2-unipotent quotient U_2 of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}},b})_{\mathbb{Q}_p}$. The pro-universal enveloping algebra of the latter can be described using the theory of Malcev completions (see [Qui69, Appendix A]), as we now recall. We will use this approach, following [BD18, Section 5], to construct A_Z. We first define

$$\mathbb{Z}_p[[\pi_1^{\text{ét},(p)}(X_{\overline{\mathbb{Q}}},b)_{\mathbb{Q}_p}]] := \varprojlim \mathbb{Z}_p[\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}},b)]/N\,,$$

where the limit is over all group algebras of finite quotients of p-power order [Qui69, Appendix A]. Letting I denote the augmentation ideal of $\mathbb{Q}_p \otimes \mathbb{Z}_p[[\pi_1^{\text{ét},(p)}(X_{\overline{\mathbb{Q}}},b)_{\mathbb{Q}_p}]]$, we define, for $n \geq 1$, the algebra

$$A_n := A_n(b) := \mathbb{Q}_p \otimes \mathbb{Z}_p[[\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}},b)_{\mathbb{Q}_p}]]/I^{n+1}\,.$$

Then A_n is a quotient of the enveloping algebra of the maximal n-unipotent quotient U_n of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}},b})_{\mathbb{Q}_p}$. More precisely, the limit of the algebras A_n is (isomorphic to) the pro-universal enveloping algebra of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}},b)_{\mathbb{Q}_p}$, see [CK10, §2]. For $n = 1, 2$, we can describe A_n as follows:

$$1 \to V \to A_1 \to \mathbb{Q}_p \to 1,$$

and, similar to §4.4, there is an exact sequence[18]

$$1 \to \operatorname{coker}(\mathbb{Q}_p(1) \xrightarrow{\cup^*} V^{\otimes 2}) \to A_2 \to A_1 \to 1\,, \tag{58}$$

coming from the isomorphism $I^2/I^3 \simeq \operatorname{coker}(\mathbb{Q}_p(1) \xrightarrow{\cup^*} V^{\otimes 2})$. Following the argument in Lemma 4.18, we can use Z to define a quotient of A_2 as follows.

Definition 5.11. The representation $A_Z := A_Z(b)$ is the pushout of A_2 by

$$\mathrm{cl}_Z^*\colon \operatorname{coker}(\mathbb{Q}_p(1) \xrightarrow{\cup^*} V^{\otimes 2}) \to \mathbb{Q}_p(1).$$

For all n, the action of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}, b)_{\mathbb{Q}_p}$ on $\mathbb{Z}_p[[\pi_1^{\text{ét},(p)}(X_{\overline{\mathbb{Q}}}, b)_{\mathbb{Q}_p}]]$ induces a Galois-equivariant action of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}, b)_{\mathbb{Q}_p}$ on A_n which factors through U_n. This induces a faithful Galois-equivariant left action of U on A_Z; the action is unipotent with respect to the I-adic filtration by construction. In fact, the I-adic filtration gives A_Z the structure of a crystalline mixed extension of G_T-representations with graded pieces $\mathbb{Q}_p, V, \mathbb{Q}_p(1)$.

Via the action of U on A_Z, we can now construct a mixed extension with graded pieces $\mathbb{Q}_p, V, \mathbb{Q}_p(1)$ from a given torsor $P \in \mathrm{Sel}(U)$, and hence from a point $x \in X(\mathbb{Q})$, by twisting A_Z (see Appendix A).

Definition 5.12. For $P \in \mathrm{H}^1(G_T, U)$ we define

$$\tau(P) := P \times_U A_Z.$$

When $x \in X(\mathbb{Q})$ and P is the path torsor $P(b,x) := \pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}, b, x)$, then we denote $A(x) := \tau(P)$.

The most important features of the twisting map τ are summarized in the following lemma. Since we want to focus on computational methods, we refer to [BD18, §5.1] and [BD21, §3.3] for proofs of these assertions.

Lemma 5.13. *Let $P \in \mathrm{H}^1(G_T, U)$. Then we have the following:*

(i) The representation $\tau(P)$ is a mixed extension of G_T-representations with graded pieces $\mathbb{Q}_p, V, \mathbb{Q}_p(1)$.
(ii) The map τ is injective.
(iii) If P is crystalline at p, then $\tau(P)$ is crystalline at p as well.
(iv) We have $\pi_1(\tau(P)) = P \times_U A_1$ and $\pi_2(\tau(P)) = P \times_U IA_Z$.

Definition 5.14. For a prime v, and $P \in \mathrm{H}^1(G_v, U)$ we define $\tau_v(P) := P \times_U A_Z$ exactly as in the global case. We will also write $A(x) := \tau_v(P(b,x))$ for $P(b,x) = \pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}, b, x)$ and $x \in X(\mathbb{Q}_v)$.

Remark 5.15. Lemma 5.13 (i), (ii) and (iv) remain valid for τ_v. If $P \in \mathrm{H}^1(G_p, U)$ is crystalline, then so is $\tau_p(P)$.

For our algorithm, we need to describe $\pi_i(A(x))$ explicitly for $i = 1, 2$ and $x \in X(\mathbb{Q})$. See [BD18, §5.2] and [BD21, Lemma 3.5] for proofs and more details. We find that on $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$, we have

$$\pi_1(A(x)) = \log([(x) - (b)])$$

(similar to §2.3), but $\pi_2(A(x))$ is more difficult to describe, since it depends on Z. In $\mathrm{Ext}^1(V, \mathbb{Q}_p(1))$ we have

$$[P(b,x) \times_U IA_Z] = E_Z(\pi_1(A(x))) + [IA_Z],$$

[18] Note that $\operatorname{coker}(\mathbb{Q}_p(1) \xrightarrow{\cup^*} V^{\otimes 2}) \simeq \operatorname{coker}(\mathbb{Q}_p(1) \xrightarrow{\cup^*} \wedge^2 V) \oplus \mathrm{Sym}^2 V$; in §4.4 there is no $\mathrm{Sym}^2 V$ summand.

where E_Z is the endomorphism of $\mathrm{H}^1_f(G_T, V)$ induced by Z. So let $c_Z \in \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ be the constant functional corresponding to $[IA_Z]$. This is a p-adic logarithm of the Chow-Heegner point associated to Z, see [BDM+19, Remarks 3.11 and 5.6] and [DRS12]. Then, in $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$, we find

$$\pi_2(A(x)) = E_Z(\log([(x) - (b)])) + c_Z\,, \tag{59}$$

where, by abuse of notation, E_Z denotes the endomorphism of $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ induced by Z.

In practice, we read off $\pi_i(A(x))$ directly from our explicit description of $A(x)$, see (81) below.

5.2. **Algorithms for the local height at p.** To compute $X(\mathbb{Q}_p)_U$, we need to explicitly compute $h_p(A(x))$. This means that we need to compute a nice correspondence Z and write the locally analytic function

$$X(\mathbb{Q}_p) \to \mathbb{Q}_p\,; \quad x \mapsto h_p(A(x))$$

as a power series on every residue disk of $X(\mathbb{Q}_p)$. In this section, we explain how this can be done. We will follow [BDM+19] quite closely.

By Proposition 3.13 we have the formula

$$h_p(A(x)) = \chi_p(\gamma_\phi - \gamma_{\mathrm{Fil}} - \beta_\phi^{\intercal} \cdot s_1(\alpha_\phi) - \beta_{\mathrm{Fil}}^{\intercal} s_2(\alpha_\phi)), \tag{60}$$

for the local height of the mixed extension $A(x)$. As explained in §(3.3.2), the quantities on the right hand side only depend on the filtered ϕ-module $D_{\mathrm{cris}}(A(x))$. In particular, we do not need to construct the representation $A(x)$ explicitly.

Thus the goal of this section is a construction and an explicit description of $D_{\mathrm{cris}}(A(x))$. Briefly, we will use the de Rham realization of A_Z: this is a filtered connection $\mathcal{A}_Z$ with Frobenius structure (more precisely, a unipotent isocrystal), and Olsson's comparison theorem [Ols11, Theorem 1.4] then implies the following isomorphism of filtered ϕ-modules:

Lemma 5.16 ([BDM+19, Lemma 5.4])**.** *We have*

$$D_{\mathrm{cris}}(A(x)) = x^*\,\mathcal{A}_Z\,, \quad \text{for all } x \in X(\mathbb{Q}_p).$$

Hence it suffices to construct the filtration and Frobenius structure of $x^*\,\mathcal{A}_Z$. The idea is to follow the construction of U_Z and A_Z via pushout given in the proof of Lemma 4.18 and in Definition 5.11, respectively. In our setting, the role of the maximal n-unipotent quotient U_2 and the algebra A_2 is played by the universal filtered connection $\mathcal{A}_2^{\mathrm{dR}}$, which we introduce below. The universal properties enjoyed by $\mathcal{A}_2^{\mathrm{dR}}$ then allow us to determine the Hodge filtration and Frobenius structure of $\mathcal{A}_Z$ uniquely. The Hodge filtration is determined by the Hodge filtration on its graded pieces, as well as its global nature: that starting with an affine piece Y, it extends nicely to X. The Frobenius structure is determined by its action on the unit vector [BDM+19, Lemma 5.2], which is essentially an initial condition for a p-adic differential equation that we can extend via parallel transport.

Following [KL, §8.3], we denote by $\mathrm{Un}^{\mathrm{dR}}(X)$ the category of unipotent vector bundles with connection on X. This is a neutral Tannakian category, i.e. a rigid abelian tensor category with a fiber functor b^*. Then $\mathrm{Un}^{\mathrm{dR}}(X)$ is unipotent, in the sense that its Tannakian fundamental group is the pro-unipotent group scheme $\pi_1^{\mathrm{dR}}(X, b)$. For background on Tannakian categories, see [DM82]. In particular, $\mathrm{Un}^{\mathrm{dR}}(X)$ is equivalent to the category of finite-dimensional K-representations of $\pi_1^{\mathrm{dR}}(X, b)$.

In analogy with the discussion in §5.1, we denote, for each $n \geq 1$, by $A_n^{\mathrm{dR}}(b)$ the quotient of the universal enveloping algebra of $\mathrm{Lie}(\pi_1^{\mathrm{dR}}(X, b))$ by the $(n+1)$th power of its augmentation ideal. For $x \in X(\mathbb{Q}_p)$, we have associated path torsors

$$A_n^{\mathrm{dR}}(b, x) := A_n^{\mathrm{dR}}(b) \times_{\pi_1^{\mathrm{dR}}(X,b)} \pi_1^{\mathrm{dR}}(X; b, x)\,, \tag{61}$$

which will be used in the description of the Frobenius structure in §5.2.3.

The theory of universal objects in unipotent Tannakian categories (see [BDM+19, Appendix A.1] and [BD21, Appendix A]) shows that there is a universal n-step unipotent object (see [BDM+19, Definition A.2, Lemma A.3])

$$\mathcal{A}_n^{\mathrm{dR}} := \mathcal{A}_n^{\mathrm{dR}}(b)$$

associated to the $\pi_1^{\mathrm{dR}}(X,b)$-representation $A_n^{\mathrm{dR}}(b)$. In other words, if $\mathcal{V}$ is an n-step unipotent connection on X and $v \in b^*\mathcal{V}$, then there is a unique morphism of connections $f\colon \mathcal{A}_n^{\mathrm{dR}} \to \mathcal{V}$ such that $b^*(f)(e_n) = v$ (e_n being the unit element). See also [Kim09, p. 98–100], and [KL, §8.3.1], where the projective system of the $\mathcal{A}_n^{\mathrm{dR}}$ is discussed.

As in [KL, §8.3.1], we can characterize the *Hodge filtration* $\mathcal{F}^\bullet$ on $\mathcal{A}_n^{\mathrm{dR}}$ as the unique descending filtration $\mathcal{F}^\bullet$ on $\mathcal{A}_n^{\mathrm{dR}}(b)$ such that Griffiths transversality holds and such that the following two properties are satisfied:

(HF1) Endow $V_{\mathrm{dR}}^{\otimes n} \otimes \mathcal{O}_X$ with the filtration induced by the Hodge filtration on $V_{\mathrm{dR}}^{\otimes n}$. Then the sequence of connections

$$V_{\mathrm{dR}}^{\otimes n} \otimes \mathcal{O}_X \to \mathcal{A}_n^{\mathrm{dR}}(b) \to \mathcal{A}_{n-1}^{\mathrm{dR}}(b) \to 0.$$

respects the filtrations.

(HF2) $1 \in \mathcal{F}^0 A_n^{\mathrm{dR}}(b)$.

For details, see [BD21, §6.3], [BDM+19, §4.3]. In particular, the map $V_{\mathrm{dR}}^{\otimes n} \otimes \mathcal{O}_X \to \mathcal{A}_n^{\mathrm{dR}}(b)$ is discussed before Theorem 4.5 in [BDM+19]. The uniqueness follows from [Had11].

In analogy with (58), we obtain an exact sequence of filtered vector bundles

$$0 \to \operatorname{coker}(\mathrm{H}^2_{\mathrm{dR}}(X)^* \xrightarrow{\cup^*} V_{\mathrm{dR}}^{\otimes 2}) \otimes \mathcal{O}(X) \to \mathcal{A}_2^{\mathrm{dR}} \to \mathcal{A}_1^{\mathrm{dR}} \to 0.$$

Recall the discussion of nice correspondences in §4.4; the statements there have natural de Rham analogues. We will denote the Tate class in $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}) \otimes \mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})$ induced by a nice correspondence Z also by Z; in analogy with (51), Z induces a map

$$\operatorname{coker}(\mathrm{H}^2_{\mathrm{dR}}(X)^* \xrightarrow{\cup^*} V_{\mathrm{dR}}^{\otimes 2}) \to \mathbb{Q}(1). \tag{62}$$

We denote by $\mathcal{A}_Z := \mathcal{A}_Z(b)$ the pushout of $\mathcal{A}_2^{\mathrm{dR}}$ by the map (62), and we obtain a connection with a filtration induced by the Hodge filtration on $\mathcal{A}_2^{\mathrm{dR}}$, satisfying $\mathcal{F}^1 \mathcal{A}_Z = 0$ and $\mathcal{F}^{-1} \mathcal{A}_Z = \mathcal{A}_Z$; thus Griffiths transversality is trivially satisfied. See [BDM+19, §4.4] for details. In particular, computing the Hodge filtration on $\mathcal{A}_Z$ means computing $\mathcal{F}^1 \mathcal{A}_Z$ explicitly.

5.2.1. *Computing the correspondence Z on $H^1_{\mathrm{dR}}(X/\mathbb{Q})$.* It can be quite difficult to geometrically compute a correspondence Z on X that is nice in the sense of Definition 4.19. For our algorithms, we will in fact need the action of Z on $H^1_{\mathrm{dR}}(X/\mathbb{Q})$ rather than a geometric description. When X is a modular curve, we can use a Hecke correspondence, as follows: We choose an auxiliary prime q such that the matrix of the Hecke operator T_q is computed with respect to our choice of basis of $H^1_{\mathrm{dR}}(X/\mathbb{Q})$, via the Frobenius matrix F, computed using Tuitman's algorithm (Algorithm 1.53) and the Eichler–Shimura relation:

$$T_q = F + qF^{-1}.$$

This q is chosen so that T_q generates the Hecke algebra. We then construct (the action of) a nice correspondence Z as a nontrivial polynomial in T_q of degree at most $\rho(J) - 1$; to ensure that Z is nice, it suffices that the corresponding matrix has trace 0.

Remark 5.17. For most quadratic Chabauty examples that have been considered, it has sufficed to use a power of T_q. Nevertheless, constructing a polynomial in powers of T_q was a crucial part of the computation of rational points on $X_{\mathrm{ns}}^+(17)$, as in Example 5.33.

In practice, we require that the basis of $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})$ is symplectic, since this is currently assumed in other parts of our algorithm. We prefer to work with the prime $q = p$, since otherwise there are issues with provable correctness.

5.2.2. *Computing the Hodge filtration.* We first fix an affine open $Y \subseteq X$ with base point $b \in Y(\mathbb{Q}) \subseteq X(\mathbb{Q})$ and determine $\mathcal{A}_Z|_Y$. This is easier than working directly on X, since unipotent vector bundles on Y are trivial.

Let $\omega_0, \dots, \omega_{2g-1} \in \mathrm{H}^0(Y_{\overline{\mathbb{Q}}}, \Omega^1)$ such that the ω_i extend to differentials on X with residue 0 whose classes form a symplectic basis of $\mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}})$ with respect to the cup product pairing. We also assume that $\omega_0, \dots, \omega_{g-1}$ form a basis of $\mathrm{H}^0(X_{\overline{\mathbb{Q}}}, \Omega^1)$. Suppose that $\#(X \setminus Y)(\overline{\mathbb{Q}}) = d$ and pick

$$\omega_{2g}, \dots, \omega_{2g+d-2} \in \mathrm{H}^0(Y_{\overline{\mathbb{Q}}}, \Omega^1)$$

with simple poles on X such that the classes of $\omega_0, \dots, \omega_{2g+d-2}$ form a basis of $\mathrm{H}^1_{\mathrm{dR}}(Y)$. Let $T_0, \dots, T_{2g+d-2}$ be the basis of $V_{\mathrm{dR}}(Y) = \mathrm{H}^1_{\mathrm{dR}}(Y)^*$ dual to $\omega_0, \dots, \omega_{2g+d-2}$.

We first discuss $\mathcal{A}_2^{\mathrm{dR}}$. For any $n \geq 1$, the connection $\mathcal{A}_n^{\mathrm{dR}}(X)|_Y$ can be obtained as a quotient of an n-step unipotent bundle $\mathcal{A}_n^{\mathrm{dR}}(Y)$ with connection, satisfying a universal property due to Kim [Kim09, p. 99] (see [BDM+19, Theorem 4.2]), such that $\mathcal{A}_n^{\mathrm{dR}}(X)|_Y$ is the maximal quotient of $\mathcal{A}_n^{\mathrm{dR}}(Y)$ which extends to a connection on X without log singularities by [BDM+19, Corollary 4.4]. We can describe $\mathcal{A}_n^{\mathrm{dR}}(Y)$ explicitly as

$$\mathcal{A}_n^{\mathrm{dR}}(Y) = \left(\bigoplus_{i=0}^{n} V_{\mathrm{dR}}(Y)^{\otimes i} \otimes \mathcal{O}_Y, \nabla_n \right), \quad \text{where } \nabla_n(v \otimes 1) = \sum_{i=0}^{2g+d-2} -(T_i \otimes v) \otimes \omega_i \,. \tag{63}$$

Since $\mathcal{A}_Z$ is a quotient of $\mathcal{A}_2^{\mathrm{dR}}$ via Z, we can use this explicit description and the unique extension of $\mathcal{A}_2^{\mathrm{dR}}(Y)$ to X to describe $\mathcal{A}_Z$ explicitly. Namely, we can trivialize

$$s_0 \colon (\mathbb{Q} \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}(1)) \otimes \mathcal{O}_Y \xrightarrow{\sim} \mathcal{A}_Z|_Y$$

such that

$$s_0^{-1} \nabla s_0 = \mathrm{d} + \Lambda \tag{64}$$

describes ∇ with respect to this trivialization, where

$$\Lambda = - \begin{pmatrix} 0 & 0 & 0 \\ \vec{\omega} & 0 & 0 \\ \eta & \vec{\omega}^{\intercal} Z & 0 \end{pmatrix} \tag{65}$$

for some differential $\eta \in \mathrm{H}^0(Y_{\overline{\mathbb{Q}}}, \Omega^1)$ with simple poles on X. In (65) and in the following, we write

$$\vec{\omega} = \{\omega_0, \dots, \omega_{2g-1}\},$$

and we denote by Z the matrix of the correspondence Z on $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})$ with respect to the basis $\vec{\omega}$, acting on column vectors. We also use block matrix notation with respect to the basis

$$\{1, T_0, \dots, T_{2g-1}, S\},$$

where S generates $\mathbb{Q}(1)$.

We now discuss how to compute the trivialization s_0, thus giving an explicit description of $\mathcal{A}_Z|_Y$. By the above, it suffices to compute the differential η. In fact, s_0 and η are not uniquely determined by the above, but as explained in [BDM+19, Remark 4.9, Lemma 4.10], there is a unique such η in the span of $\{\omega_{2g}, \dots, \omega_{2g+d-2}\}$. Let $L/\mathbb{Q}$ be a finite extension over which all points of $X \setminus Y$ are defined. By the proof of [BDM+19, Lemma 4.10], we may compute η explicitly as follows:

- For all $x \in X$, let $\vec{\omega}_x$ be a vector of Laurent series over L, containing local expansions of $\vec{\omega}$ around x. Choose a vector $\vec{\Omega}_x$ such that $\mathrm{d}\vec{\Omega}_x = -\vec{\omega_x}$.
- Solve for the unique $\eta = \sum_{i=2g}^{2g+d-2} e_i\omega_i$ such that

$$\mathrm{Res}(d\vec{\Omega}_x^{\intercal} Z \vec{\Omega}_x - \eta) = 0 \quad \text{for all } x \in X \setminus Y\,. \tag{66}$$

The proof uses a trivialization of $\mathcal{A}_Z$ in a formal neighborhood of x and a unipotent gauge transformation. In order to compute η in practice, it suffices to compute local coordinates at all $x \in X \setminus Y$ and to solve for the e_i by computing the residues necessary to solve the system of equations arising from (66).

Having an explicit description of $\mathcal{A}_Z|_Y$, we now discuss how to compute the Hodge filtration on $\mathcal{F}^0(\mathcal{A}_Z)$, the Hodge filtration on its graded pieces, and the fact that it is uniquely characterized by conditions (HF1) and (HF2). As in Example 3.10(5), there is a filtration on $(\mathbb{Q} \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}(1)) \otimes \mathcal{O}_Y$ given by

$$\begin{aligned} \mathcal{F}^{-1} &= (\mathbb{Q} \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}(1)) \otimes \mathcal{O}_Y \\ \mathcal{F}^{0} &= (\mathbb{Q} \oplus \mathcal{F}^0 V_{\mathrm{dR}}) \otimes \mathcal{O}_Y \\ \mathcal{F}^{1} &= 0. \end{aligned} \tag{67}$$

Our goal is to find a filtration-respecting trivialization

$$s^{\mathrm{Fil}} \colon (\mathbb{Q} \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}(1)) \otimes \mathcal{O}_Y \xrightarrow{\sim} \mathcal{A}_Z|_Y \tag{68}$$

(or, more precisely, its restriction to $\mathcal{F}^0 = (\mathbb{Q} \oplus \mathcal{F}^0 V_{\mathrm{dR}})$). For each $x \in X \setminus Y$, we choose a local coordinate t_x at x, and we define $g_x \in L((t_x))$ by requiring

$$\mathrm{d}g_x = \vec{\Omega}_x^{\intercal} Z\, \mathrm{d}\vec{\Omega}_x - \eta. \tag{69}$$

Let

$$N = (0_g, 1_g)^{\intercal} \in M_{2g \times g}(\mathbb{Q})$$

and define

$$\gamma_{\mathrm{Fil}} \in \mathcal{O}_Y, \qquad b_{\mathrm{Fil}} = (b_g, \dots, b_{2g-1})^{\intercal} \in \mathbb{Q}^g$$

by the requirement that

$$\gamma_{\mathrm{Fil}}(b) = 0$$

and for all $x \in X \setminus Y$:

$$g_x + \gamma_{\mathrm{Fil}} - b_{\mathrm{Fil}}^{\intercal} N^{\intercal} \vec{\Omega}_x - \vec{\Omega}_x^{\intercal} Z N N^{\intercal} \vec{\Omega}_x \in L[[t_x]]\,.$$

Remark 5.18. If X is hyperelliptic, then $\eta = 0$ and $b_{\mathrm{Fil}} = (0, \dots, 0)^{\intercal}$ by [BD21, Lemma 6.5].

We can finally describe the Hodge filtration $\mathcal{F}^0 \mathcal{A}_Z$ in terms of b_{Fil} and γ_{Fil} as follows:

Theorem 5.19 ([BDM+19, Theorem 4.11]). *We can choose s^{Fil} in (68) such that the restriction of $s_0^{-1} s^{\mathrm{Fil}}$ to $(\mathbb{Q} \oplus \mathcal{F}^0 V_{\mathrm{dR}}) \otimes \mathcal{O}_Y$ is given by the $(2g+2) \times (g+1)$ matrix*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & N \\ \gamma_{\mathrm{Fil}} & b_{\mathrm{Fil}}^{\intercal} \end{pmatrix}.$$

The proof of Theorem 5.19 proceeds in two steps: First show directly that the bundle $\mathcal{H} := s_0 \circ H(\mathbb{Q} \oplus \mathcal{F}^0 V_{\mathrm{dR}})$ extends to a sub-bundle of $\mathcal{A}_Z$. Then check that the requirements ((HF1)) and ((HF2)) are satisfied, which follows easily from the shape of H.

From this result, we obtain the following algorithm, where we assume that the matrix Z and the differentials ω_i are given.

Algorithm 5.20 (The Hodge filtration on $\mathcal{A}_Z$)**.**

(1) Compute local coordinates t_x at each $x \in X \setminus Y$.
(2) At each $x \in X \setminus Y$, expand $\vec{\omega}$ into a vector $\vec{\omega}_x$ of Laurent series.
(3) Compute a vector $\vec{\Omega}_x$ that satisfies $d\vec{\Omega}_x = -\vec{\omega}_x$.
(4) Solve for η as the unique linear combination of $\omega_{2g}, \dots, \omega_{2g+d-2}$ such that (66) is satisfied.
(5) Solve the system of equations for g_x such that $dg_x = \vec{\Omega}_x^{\intercal} Z d\vec{\Omega}_x - \eta$.
(6) Compute the vector of constants $b_{\mathrm{Fil}} = (b_g, \dots, b_{2g-1}) \in \mathbb{Q}^g$ and the function γ_{Fil} characterized by $\gamma_{\mathrm{Fil}}(b) = 0$ and

$$g_x + \gamma_{\mathrm{Fil}} - b_{\mathrm{Fil}}^{\intercal} N^{\intercal} \vec{\Omega}_x - \vec{\Omega}_x^{\intercal} Z N N^{\intercal} \vec{\Omega}_x \in L[[t_x]],$$

where $N = (0_g, 1_g)^{\intercal} \in M_{2g \times g}(\mathbb{Q})$. Set $\beta_{\mathrm{Fil}} = (0, \dots, 0, b_g, \dots, b_{2g-1})^{\intercal}$.

By Lemma 5.16, the Hodge filtration on $\mathcal{A}_Z$ gives us the Hodge filtration on $D_{\mathrm{cris}}(A(x))$ for any $x \in X(\mathbb{Q}_p)$. More precisely, we find:

Corollary 5.21. *The vector β_{Fil} and the function γ_{Fil} computed in Algorithm 5.20 are the same as β_{Fil} and γ_{Fil} in Proposition 3.13 for $E_p = A(x)$.*

In Algorithm 5.20, the main task is to compute the principal parts of the Laurent series that appear in the various systems of equations. It suffices to compute to t_x-adic precision $t_x^{d_x}$, where d_x is the order of the largest pole occurring. Note that if $[L : \mathbb{Q}]$ is large, then the computation of the Hodge filtration can become quite expensive. Hence it is often useful to look for a model of X such that $[L : \mathbb{Q}]$ is small. One could potentially avoid this issue by working p-adically. However, `Magma's` function field functionality, which we use, is limited to exact fields. Moreover, one would then need a p-adic precision analysis.

5.2.3. *Computing the Frobenius structure.* Recall that our goal is to explicitly compute the filtered ϕ-module $D_{\mathrm{cris}}(A(x))$ and that by Lemma 5.16, we get an isomorphism of filtered ϕ-modules

$$D_{\mathrm{cris}}(A(x)) = x^* \mathcal{A}_Z$$

for all $x \in X(\mathbb{Q}_p)$. We now discuss how to endow the base change of the filtered connection $\mathcal{A}_Z$ introduced above to $\mathbb{Q}_p$ with a Frobenius structure and how to compute the latter. The Frobenius structure is defined on a unipotent isocrystal $\mathcal{A}_Z^{\mathrm{rig}}(\overline{b})$, obtained via analytification.

We first discuss unipotent isocrystals. To simplify the exposition, we focus on the affinoid setting, following Besser's notes [Bes12, §1.5]. Let A be an affinoid algebra with good reduction over K, a complete discrete valuation field of characteristic 0, and let $A^{\dagger}$ be its weak completion, as in Chapter 1.3. Let $\overline{A} = A^{\dagger}/\pi$ where π is a uniformizer of R, the ring of integers of K.

Definition 5.22. A *unipotent isocrystal* on $\overline{A}$ is an $A^{\dagger}$-module M together with an (integrable) connection

$$\nabla \colon M \to M \otimes_{A^{\dagger}} \Omega^1(\otimes K)$$

that is an iterated extension of trivial connections, where the trivial connection is $\mathbb{1} = (A^\dagger, d)$. A *morphism* of unipotent isocrystals is a map of $A^\dagger$-modules that is horizontal (i.e. commutes with connection). Let $\mathrm{Un}(\overline{A})$ denote the category of unipotent isocrystals on $\overline{A}$.

Any unipotent isocrystal is overconvergent. Moreover, the category $\mathrm{Un}(\overline{A})$ is a neutral Tannakian category, with fiber functor $\overline{b}^*$ defined by sending (M, ∇) to the vector space of sections of M on the residue disk of $\overline{b}$ that ∇ vanishes in, where $\overline{b} \in \mathrm{Spec}(\overline{A})(k)$ and k is the residue field. The Tannakian fundamental group $\pi^1(\mathrm{Un}(\overline{A}), \overline{b}^*))$ is a unipotent affine group scheme. A k-linear Frobenius on $\overline{A}$ that fixes $\overline{b}$ induces an isomorphism of $\pi^1(\mathrm{Un}(\overline{A}), \overline{b}^*))$. Similarly, if $\overline{b}, \overline{x} \in \mathrm{Spec}(\overline{A})(k)$, we obtain a path torsor $\pi^1(\mathrm{Un}(\overline{A}), \overline{b}^*, \overline{x}^*)$, which is a homogeneous space for $\pi^1(\mathrm{Un}(\overline{A}), \overline{b}^*))$ and a k-Frobenius that fixes $\overline{b}$ and $\overline{x}$ induces an action on $\pi^1(\mathrm{Un}(\overline{A}), \overline{b}^*, \overline{x}^*)$.

Besser used unipotent isocrystals to give a more conceptual approach to iterated Coleman integrals than introduced in §1.5. We will not phrase our results in this language, but we briefly discuss it here. The rough idea is that an iterated Coleman integral

$$\int \omega_n \dots \omega_1$$

is the y_n-coordinate of a solution to the unipotent system of p-adic differential equations

$$d\vec{y} = \Omega \vec{y}, \text{ where } \Omega := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \omega_1 & 0 & \cdots & 0 & 0 \\ 0 & \omega_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \omega_n & 0 \end{pmatrix}, \tag{70}$$

with $y_0 = 1$. This system has a local solution in each residue disk. Recall that in §1, we used the change-of-variables formula with respect to Frobenius to analytically continue tiny integrals from the residue disk of $\overline{b}$ to that of $\overline{x}$ in a unique way. In the unipotent isocrystal setting, Besser showed that there is a unique Frobenius-invariant path in $\pi^1(\mathrm{Un}(\overline{A}), \overline{b}^*, \overline{x}^*)$, which allows us to analytically continue local solutions to the system above. It is not hard to see that for the unipotent isocrystal corresponding to (70), the two notions of analytic continuation coincide; see [Bes12, p. 19].

More generally, Besser defines an *abstract Coleman function* on $A^\dagger$ to be a tuple $((M, \nabla), \vec{y}, s)$, where (M, ∇) is a unipotent isocrystal on $\overline{A}$, $\vec{y} = (y_{\overline{x}})_{\overline{x}}$ a collection of sections to M, one for each residue disc, that are horizontal with respect to ∇ and compatible with the Frobenius-invariant paths, and $s \in \mathrm{Hom}(M, A^\dagger)$. The *Coleman function* corresponding to an abstract Coleman function $((M, \nabla), \vec{y}, s)$ is then the function given on the residue disk of $\overline{x}$ by $s \circ y_{\overline{x}}$.

Example 5.23. Let's reinterpret a single Coleman integral $\int \omega$ in this setting. It corresponds to the system $d(y_0, y_1) = (0, \omega y_0)$, with $y_0 = 1$, which gives rise to the unipotent isocrystal $(A^\dagger \oplus A^\dagger, \nabla)$, where

$$\nabla(y_1, y_2) = \left(d - \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix} \right) (y_1, y_2) = (dy_1, dy_2 - \omega y_1) .$$

Define sections $y_{\overline{x}}$ by fixing a local solution $(1, y_2)$ in the disk of a base point $\overline{b}$ and analytically continuing it using the Frobenius-invariant path to all residue disks. Let $s(y_1, y_2) = y_2$. Then the Coleman function corresponding to the abstract Coleman function $((A^\dagger \oplus A^\dagger, \nabla), \vec{y}, s)$ is precisely the Coleman integral $\int \omega$. See [Bes12, Examples 13, 30] for details.

One obtains iterated Coleman integrals using the unipotent isocrystal arising from (70) in a similar way. Besser defines, more generally, abstract Coleman functions with values in a sheaf and shows that these form a K-algebra. In fact, Coleman functions are locally analytic and satisfy an identity principle. The Chabauty and quadratic Chabauty functions discussed in these notes are all Coleman functions. Moreover, the Chabauty–Kim functions j_U^{dR} are Coleman; in fact their coordinates are iterated Coleman integrals.

For quadratic Chabauty, we need to use the category $\mathrm{Un}(X_{\mathbb{F}_p})$ of unipotent isocrystals on $X_{\mathbb{F}_p}$ rather than on affinoids. These were constructed by Berthelot in [Ber96] using the notion of rigid triples. See [Bes02], [BDM+19, Appendix A.2] and [CLS99]. In this case, the underlying module is a locally free $j^\dagger\mathcal{O}_{]Y_{\mathbb{F}_p}[}$-module, where $j^\dagger\mathcal{O}_{]Y_{\mathbb{F}_p}[}$ is Berthelot's overconvergent structure sheaf on the tube $]Y_{\mathbb{F}_p}[\subset X(\mathbb{Q}_p)$ (see [Ber96, §2.1.1.3]), which consists of all points that reduce to a point in $Y_{\mathbb{F}_p}$. To simplify notation, we denote

$$R^\dagger := \mathrm{H}^0\left(]Y_{\mathbb{F}_p}[, j^\dagger\mathcal{O}_{]Y_{\mathbb{F}_p}[}\right).$$

In practice, this will be the same ring $R^\dagger$ of overconvergent functions defined before Definition 1.44.

As in the affinoid setting, there is a Frobenius action (in fact an auto-equivalence) on $\mathrm{Un}(X_{\mathbb{F}_p})$ and there are induced Frobenius actions on the path torsors $\pi^1(\mathrm{Un}(\overline{A}), \overline{b}^*, \overline{x}^*)$. The n-step unipotent quotients $A_n^{\mathrm{rig}}(\overline{b})$ and $A_n^{\mathrm{rig}}(\overline{b}, \overline{x})$ are constructed as in the de Rham case. These also inherit Frobenius actions, and in particular, the universal n-step unipotent object

$$\mathcal{A}_n^{\mathrm{rig}} := \mathcal{A}_n^{\mathrm{rig}}(\overline{b})$$

is canonically isomorphic to its pullback via ϕ, giving rise to the *Frobenius structure*

$$\Phi_n \colon \phi^*\,\mathcal{A}_n^{\mathrm{rig}} \to^{\sim} \mathcal{A}_n^{\mathrm{rig}}\,.$$

By [BDM+19, Lemma 5.2], Φ_n is the unique morphism that fixes 1 in the fiber above $\overline{b}$, and we may use this to determine Φ_n explicitly. As usual, we can take a quotient of A_2^{rig} by our nice correspondence Z to obtain A_Z^{rig}, $A_Z^{\mathrm{rig}}(\overline{b}, \overline{x})$ and $\mathcal{A}_Z^{\mathrm{rig}}$ and an induced Frobenius structure

$$\Phi_Z \colon \phi^*\,\mathcal{A}_Z^{\mathrm{rig}} \to^{\sim} \mathcal{A}_Z^{\mathrm{rig}}\,.$$

Recall that we want a Frobenius structure on $\mathcal{A}_Z$ and its pullbacks $x^*\,\mathcal{A}_Z$. Chiarellotto and Le Stum have shown (see Theorem 5.25 below) that analytification gives an equivalence between the categories $\mathrm{Un}^{\mathrm{dR}}(X_{\mathbb{Q}_p})$ and $\mathrm{Un}(X_{\mathbb{F}_p})$, compatible with fiber functors. In particular, we have $\mathcal{A}_Z^{an} = \mathcal{A}_Z^{\mathrm{rig}}$.

To make the Frobenius structure explicit, we restrict to Y, where all our bundles become trivial, and use the explicit description (64) of the connection on $\mathcal{A}_Z\,|_Y$. Accordingly, the connection on $\phi^*(\mathcal{A}_Z^{\mathrm{rig}})|_Y$ is given with respect to s_0 by $d + \phi^*\Lambda$. Then on Y, the inverse of Φ_Z is an isomorphism of 2-unipotent connections that respects the unipotent filtrations. On the given trivializations, it satisfies $\Phi_Z^{-1} \circ (d - \Lambda) = (d - \phi^*\Lambda) \circ \Phi_Z^{-1}$, and therefore

$$(\phi^*\Lambda)\Phi_Z^{-1} + d\Phi_Z^{-1} = \Phi_Z^{-1}\Lambda\,.$$

Hence, with respect to our fixed basis $\vec{\omega}$, we are looking for a matrix $G \in (R^\dagger \otimes \mathbb{Q}_p)^{(2g+2)\times(2g+2)}$ that satisfies

$$(\phi^*\Lambda)G + dG = G\Lambda\,.$$

We denote by b_0 the Teichmüller point in the residue disk of b.

Proposition 5.24. *Define*

$$G = \begin{pmatrix} 1 & 0 & 0 \\ \vec{f} & F & 0 \\ h & \vec{g}^{\intercal} & p \end{pmatrix} \in (R^{\dagger} \otimes \mathbb{Q}_p)^{(2g+2)\times(2g+2)} \tag{71}$$

by requiring

$$\phi^* \vec{\omega} = F\vec{\omega} + d\vec{f} \quad (\text{with } \vec{f}(b_0) = \vec{0}) \tag{72}$$

$$d\vec{g}^{\intercal} = d\vec{f}^{\intercal} Z F \tag{73}$$

$$dh = \vec{\omega}^{\intercal} Z \vec{f} + d\vec{f}^{\intercal} Z \vec{f} - \vec{g}^{\intercal}\vec{\omega} + \phi^*\eta - p\eta \quad (\text{with } h(b_0) = 0). \tag{74}$$

Then G describes the inverse of the Frobenius structure Φ_Z^{-1} restricted to Y with respect to $\vec{\omega}$.

Proof. Suppose that G is a matrix (71) for some $\vec{f}, \vec{g}$ and h (note that we may assume G to have diagonal blocks $1, F$ and 1, since the G we're looking for describes a Frobenius structure). Expanding out the condition $(\phi^*\Lambda)G + dG = G\Lambda$, we find that it's equivalent to

$$\begin{aligned} \phi^*\vec{\omega} - d\vec{f} &= F\vec{\omega} \\ \phi^*\vec{\omega}^{\intercal} Z F - d\vec{g}^{\intercal} &= p\vec{\omega}^{\intercal} Z \\ \phi^*\eta + (\phi^*\vec{\omega}^{\intercal} Z)\vec{f} - dh &= \vec{g}^{\intercal}\vec{\omega} + p\eta\,. \end{aligned}$$

Using $F^{\intercal} Z F = pZ$ and the first identity, the second identity is equivalent to $d\vec{g} = -F^{\intercal} Z d\vec{f}$. Hence the matrix in the proposition satisfies $\phi^*\Lambda G + dG = G\Lambda$. The conditions on b_0 then guarantee that the universal property that Φ_Z inherits from that of Φ_2 is satisfied. □

Now we discuss how to compute G. We already know how to compute the matrix F and the vector $\vec{f} \in (R^{\dagger} \otimes \mathbb{Q}_p)^{2g}$, by applying Tuitman's algorithm (Algorithm 1.53). (Note that In Section 1, the Frobenius matrix was denoted by M (whose columns gave the Frobenius action), but we rename it here to F (and use the `Magma` convention of rows) to avoid the clash in notation; likewise $\vec{f}$ was previously denoted as $\vec{h}$.) Although we only stated this algorithm for basis differentials, it can be applied to any differential of the second kind. We want to make use of this to find $\vec{g}$ and h, so we need to find a suitable differential. One can show that

$$\xi := (\phi^*\vec{\omega}^{\intercal}) Z \vec{f} + (\phi^*\eta - p\eta) + (F^{\intercal} Z \vec{f})^{\intercal}\vec{\omega}\,. \tag{75}$$

is of the second kind.Applying Algorithm 1.53 to ξ yields $\vec{c} \in \mathbb{Q}_p^{2g}$ and $h \in R^{\dagger} \otimes \mathbb{Q}_p$ such that

$$\vec{c}^{\intercal}\vec{\omega} + dh = \xi. \tag{76}$$

Then the properties of Proposition 5.24 are satisfied for $\vec{g} := -F^{\intercal} Z \vec{f} + \vec{c}$ and $h := h - h(b_0)$.

Now that we can determine the Frobenius structure on $\mathcal{A}_Z$ explicitly, we turn to the Frobenius structure on $x^*\mathcal{A}_Z$. To this end, we use the full force of the following crucial comparison result.

Theorem 5.25 (Chiarelletto-Le Stum [CLS99])**.** *There is an equivalence of categories*

$$\mathrm{Un}^{\mathrm{dR}}(X_{\mathbb{Q}_p}) \xrightarrow{\sim} \mathrm{Un}(X_{\mathbb{F}_p})$$

given by the analytification functor $(\cdot)^{an}$.

For any $x \in X(\mathbb{Q}_p)$ with reduction $\overline{x}$, we have a canonical isomorphism of fiber functors

$$i_x : \overline{x}^* \circ (\cdot)^{an} \simeq x^*$$

such that if $x, y \in X(\mathbb{Q}_p)$ belong to the same residue disk, the canonical isomorphism $\tau_{x,y} := i_x \circ i_y^{-1}$ is given by parallel transport $T_{x,y}$ along the connection.

Theorem 5.25 induces a Frobenius structure $\phi_n(b,x)$ on $A_n^{\mathrm{dR}}(b,x)$ given by

$$\phi_n(b,x) = \tau_{b,x} \circ \phi_n(b_0,x_0) \circ \tau_{b,x}^{-1},$$

where x_0 is the Teichmüller point in the disk of x. Via Theorem 5.25, we have $\phi_n(b_0,x_0) = x_0^*\Phi_n$. The Frobenius operators that we obtain upon quotienting out by Z then satisfy

$$\phi_Z(b_0,x_0) = x_0^*\Phi_Z\,. \tag{77}$$

It remains to discuss the computation of $\tau_{b,x}$ on $A_n^{\mathrm{dR}}(b,x)$. For any $x_1,x_2 \in X(\mathbb{Q}_p)$ we define $I(x_1,x_2) \in \bigoplus_{i=0}^n V_{\mathrm{dR}}(Y)^{\otimes i}$ as

$$I(x_1,x_2) = 1 + \sum_w \int_{x_1}^{x_2} w(\omega_0,\dots,\omega_{2g+d-2})$$

where the sum is over all words w in $\{T_0,\dots,T_{2g+d-2}\}$ of length at most n, making substitution of T_i with ω_i. We will only apply this to points lying in the same residue disks, so we only need tiny iterated integrals. However, the above makes sense for arbitrary $x_1,x_2 \in X(\mathbb{Q}_p)$ using iterated Coleman integrals. By [BDM+19, §5.2.1], we obtain $\tau_{b,x}$, when considered on $A_n^{\mathrm{dR}}(Y)$ via s_0 as

$$\begin{aligned} \tau_{b,x}\colon \bigoplus_{i=0}^n V_{\mathrm{dR}}(Y)^{\otimes i} &\xrightarrow{\sim} \bigoplus_{i=0}^n V_{\mathrm{dR}}(Y)^{\otimes i} \\ v &\mapsto I(x_0,x)vI(b,b_0). \end{aligned} \tag{78}$$

By the above, we have the equality

$$\phi_Z(b,x) = \tau_{b,x} \circ \phi_Z(b_0,x_0) \circ \tau_{b,x}^{-1} \tag{79}$$

which we will use to compute $\phi_Z(b,x)$. To describe $\tau_{b,x}$ explicitly, we need $I(x_0,x)$ and $I(b,b_0)$ on $(\mathbb{Q}_p \oplus V_{\mathrm{dR}} \oplus \mathbb{Q}(1)) \otimes \mathcal{O}_Y \subset \bigoplus_{i=0}^2 V_{\mathrm{dR}}(Y)^{\otimes i}$. In this case, the matrices

$$I^{\pm}(x_1,x_2) = \begin{pmatrix} 1 & 0 & 0 \\ \int_{x_1}^{x_2} \vec{\omega} & 1 & 0 \\ \int_{x_1}^{x_2} \eta + \int_{x_1}^{x_2} \vec{\omega} + Z\vec{\omega} & \pm \int_{x_1}^{x_2} \vec{\omega}^{\intercal} Z & 1 \end{pmatrix}$$

describe $I(x_1,x_2)$ and $I(x_2,x_1)$, respectively, for $x_1,x_2 \in X(\mathbb{Q}_p)$ (see [BDM+19, §5.3.1].

Algorithm 5.26 (The Frobenius structure on $x^*\mathcal{A}_Z$)**.**

(1) Use Algorithm 1.53 to compute the matrix of Frobenius F and the vector of overconvergent function $\vec{f}$ such that $\phi^*\vec{\omega} = F\vec{\omega} + d\vec{f}$.
(2) Compute the differential ξ in (75) and apply Algorithm 1.53 to compute the vector of constants $\vec{c}$ and the overconvergent function h such that (76) holds. Then set $\vec{g} := -F^{\intercal}Z\vec{f} + \vec{c}$ and $h := h - h(b_0)$.
(3) Compute

$$M(b_0,x_0) = \begin{pmatrix} 1 & 0 & 0 \\ (I-F)^{-1}\vec{f} & 1 & 0 \\ \frac{1}{1-p}(\vec{g}^{\intercal}(I-F)^{-1}\vec{f} + h) & \vec{g}^{\intercal}(F-p)^{-1} & 1 \end{pmatrix}(x_0)\,.$$

(4) Compute the matrix $L(b,x) = I^+(x,x_0)I^-(b_0,b)$.
(5) Compute the matrix

$$s_0^{-1}(b,x) \circ s^{\phi}(b,x) = L(b,x) \cdot M(b_0,x_0) =: \begin{pmatrix} 1 & 0 & 0 \\ \vec{\alpha}_{\phi}(b,x) & 1 & 0 \\ \gamma(b,x) & \vec{\beta}_{\phi}(b,x) & 1 \end{pmatrix}$$

When applying this algorithm in practice, most of the time is usually spent on Tuitman's reduction algorithm, which is needed in various parts, and on multiplying (a large number of) power series.

To summarize, we have described algorithms to compute matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma_{\mathrm{Fil}}(b,x) & \vec{\beta}^{\mathsf{T}}_{\mathrm{Fil}}(b) & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ \vec{\alpha}_\phi(b,x) & 1 & 0 \\ \gamma_\phi(b,x) & \vec{\beta}^{\mathsf{T}}_\phi(b,x) & 1 \end{pmatrix},$$

and the entries are exactly the ingredients for our formula (60) for the local height $h_p(A(x))$.

5.3. **Algorithms for quadratic Chabauty.** We now describe how Theorem 5.2 may be used to compute $X(\mathbb{Q})$. For this, we keep the notation introduced so far in this section. We refer to [BDM+b] for more details; in fact, some parts of [BDM+b] are based on the original version of these notes. Here we are less precise than in [BDM+b], and instead focus on the main ideas. In particular, we do not discuss issues of precision; these are analyzed in [BDM+b, §4].

Algorithm 5.27.

Input:

- A modular curve $X/\mathbb{Q}$ with Mordell–Weil rank $r = g$ and $\mathrm{rk}_{\mathbb{Z}}\,\mathrm{NS}(J) > 1$, a point $b \in X(\mathbb{Q})$, and a prime p of good reduction for which the image of $J(\mathbb{Q})$ in $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ has rank g.
- A covering of X by affine opens that are birational to a planar curve cut out by an equation that is monic in one variable, has p-integral coefficients and satisfies Assumption 1.
- The action of a nice correspondence Z on $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})$, computed as in §5.2.1.

Output: The set $X(\mathbb{Q}_p)_U$, where $U = U_Z$.

(1) Write the function $x \mapsto h_p(A(x))$ as a convergent power series on every residue disk in $X(\mathbb{Q}_p)$ using Algorithm 5.20 and Algorithm 5.26.
(2) Compute the finite set S of possible values of ρ, as in §5.3.1.
(3) Compute the constants α_i, as in §5.3.2.
(4) Write the functions $x \mapsto \psi_i \circ (\pi_1, \pi_2)(A(x))$, and hence the function ρ as a convergent power series on every residue disk $D \subset X(\mathbb{Q}_p)$, solve for the points $z \in D(\mathbb{Q}_p)$ satisfying $\rho(z) \in S$ and return the union of these points for all D.

See [BDM+b, Algorithm 3.12] for a more precise version of Algorithm 5.27. In §5.4, we discuss our `Magma` implementation of this algorithm. Note the similarities to Algorithm 2.37, where some of the steps are given in more detail. We have already discussed the most difficult step, namely Step (1), the computation of the local height $h_p \circ A$, in §5.2. In practice, we can only run this step in residue disks that are good in the sense of Definition 1.48; recall that these are the disks where Tuitman's Frobenius lift is defined. Hence, we typically have to run it for several affine patches Y covering X. Sometimes, we get away with only one affine patch Y (see §5.5): if there are no small rational points in a bad disk, then we can try to show that there are none at all using the Mordell–Weil sieve, see §2.3.2. Otherwise, we can pick an affine patch such that Frobenius is defined in this disk or we can use a trick described in [BDM+19, §5.5], essentially reducing the computation of $h_p(A(x))$ for $x \in D(\mathbb{Q}_p)$ to the computation of Coleman integrals $\int_b^P \omega_i$, where $P \in D(\mathbb{Q})$.

There is no reason to expect that $X(\mathbb{Q}_p)_U = X(\mathbb{Q})$ and indeed, this is typically not the case. In practice, we first compute a set $X(\mathbb{Q})_{\mathrm{known}}$ of rational points up to some bound. If we find that $X(\mathbb{Q}_p)_U \setminus X(\mathbb{Q})_{\mathrm{known}} \neq \varnothing$, then we may try to show that these points are not rational by applying the Mordell–Weil sieve discussed in §2.3.2 similar to Step (6) of Example 2.40. See also Example 5.5

below. Alternatively, if $\rho(J) > 2$, then we can often run Algorithm 5.27 for several independent nice correspondences $Z_1, \dots, Z_{\rho(J)-1}$, and we expect that $\cap_i X(\mathbb{Q}_p)_{U_{Z_i}} = X(\mathbb{Q})$, and hence this should suffice to prove that $X(\mathbb{Q})_{\text{known}} = X(\mathbb{Q})$ (provided this is indeed the case, of course). In theory, it is also possible that $X(\mathbb{Q})_{\text{known}}$ is strictly smaller than $X(\mathbb{Q})$, though we do not expect that to happen.

Remark 5.28. As suggested by the notation, the set of points cut out by the condition $\rho(x) \in S$ in Theorem 5.2 does not depend on the choices of s or χ (note that we are working over $\mathbb{Q}$, so χ is well-defined up to a scalar multiple). See [BDM+19, Remark 3.12].

We now discuss Steps (1)–(4). Step (4) is standard, and is exactly as in Chabauty-Coleman or quadratic Chabauty for integral points, discussed in earlier sections.

5.3.1. *Local heights away from p.* Let $\ell \neq p$ be prime. We already know that by Theorem 4.3 the function

$$\text{Sel}(U) \to \mathbb{Q}_p\,; \quad P \mapsto h_\ell \circ \tau_\ell \circ \text{loc}_\ell(P) \tag{80}$$

takes values in a finite set S_ℓ, and that S_ℓ is trivial when X has potentially good reduction at ℓ. This suffices, for instance, to compute the rational points on the split Cartan modular curve $X_{\text{s}}^+(13)$, by [BDM+19, Theorem 6.6]. To apply Algorithm 5.27 for curves without everywhere potentially good reduction, we need to be able to compute S_ℓ. This is similar to, but much more difficult than Step (4) of Algorithm 2.37, discussed in detail in §2.3.1. One approach follows from work of Betts and Dogra:

Theorem 5.29 (Betts–Dogra)**.** *Let $\mathcal{X}$ be a semistable regular model of $X_{\mathbb{Q}_\ell}$ over some extension of $\mathbb{Z}_\ell$.*

(1) The functions $j_{U,\ell}$ and (80) are constant on preimages of the irreducible components of the special fiber $\mathcal{X}_\ell$ of $\mathcal{X}$.
(2) The function (80) and the set S_ℓ can be expressed in terms of harmonic analysis on the reduction graph of $\mathcal{X}$ in the sense of Zhang [Zha93].

Theorem 5.29 follows from Theorem 1.1.2, Lemma 12.1.1, and Corollary 12.1.3 of [BD19b]. See [BDM+b, Theorem 3.2] for a more precise statement.

Remark 5.30. Theorem 5.29 implies, in particular, that S_ℓ is often trivial, even when X does not have potentially good reduction at ℓ. Namely, if $X \times \mathbb{Q}_\ell$ has a semistable regular model over an extension such that the special fiber is irreducible, then $S_\ell = \{0\}$, since $j_{U,\ell}$ is constant, and vanishes in b by construction.

Example 5.31. If $X\colon y^2 = f(x)$ is hyperelliptic, $\ell > 2$ and the discriminant $\Delta(X)$ satisfies $\text{ord}_\ell(\Delta(X)) = 1$, then $S_\ell = \{0\}$.

Example 5.32 ([BDM+b, Lemma 5.2]). Let $N > 2$ be prime and let w_N be the Atkin–Lehner involution on $X_0(N)$. Then the curve $X_0^+(N) = X_0(N)/w_N$ has good reduction away from N. By work of Xue [Xue09], there is a regular semistable model (over an extension) at N whose special fiber is a projective line intersecting itself $g(X_0^+(N))$ times. Therefore the local heights at N are all trivial on $\text{Sel}(U)$, although $X_0^+(N)$ does not have potentially good reduction at N.

To make Theorem 5.29, or rather its more precise version [BDM+b, Theorem 3.2] explicit, one needs to compute the action of Z on the various pieces of the étale cohomology of $\mathcal{X}_\ell$. In general a nice correspondence will not extend to a correspondence on $\mathcal{X}$, but it does induce an action on $H_1(\Gamma, \mathbb{Q}_\ell)$, where Γ is the dual graph of $\mathcal{X}_\ell$, and on the weight -1 part of the Tate module of X, and hence an action on the Tate modules of the irreducible components of $\mathcal{X}_\ell$. In general, the computation of these actions is quite difficult. In some cases, [BDM+b, Theorem 3.2] gives enough information to conclude that

the function induced by $j_{U,\ell}$ on the reduction graph is affine linear (it is always piecewise polynomial). Together with a supply of sufficiently many rational points, this sometimes makes it possible to compute the set $\mathcal{S}_\ell$ from an overdetermined system, even if we do not know the actions of Z. See [BDM+b, §5.4] for two examples.

Example 5.33. For the modular curve $X_{\mathrm{ns}}^+(17)$, it is possible to compute the action of certain Hecke operators on Γ and the irreducible components using the action of inertia. This made it possible to construct a nice correspondence such that $S_{17} = \{0\}$, which was the main ingredient for the computation of $X_{\mathrm{ns}}^+(17)(\mathbb{Q})$ in [BDM+b, §5.5].

To compute further examples, a general algorithm to compute the above-mentioned actions of Z is required. In work in progress, Betts, Duque-Rosero, Hashimoto and Spelier use a result of Coleman–Iovita [CI99] to show how to computes the action of Z on $H_1(X, \Gamma)$ via the action on de Rham cohomology; they are working on an algorithm to compute the action on the Tate modules of irreducible components of genus > 0. Similarly, work in progress of Besser, Müller and Srinivasan expresses S_ℓ in terms of ℓ-adic double Vologodsky integrals. It would be interesting to compute S_ℓ directly using the intersection-theoretic interpretation of $h_\ell(z)$ as the local Coleman–Gross height pairing $h_\ell(z - b, D_Z(b, z))$; see Remark 5.7.

In any event, in order to compute S_ℓ, a first required step is to compute a regular semistable model. There are various software packages to compute this. Sometimes, this can be done via the `Magma` package `RegularModel` (when there is a semistable regular model over the ground field or a small extension). One can, alternatively, use the `Sage` toolbox `MCLF` [19] due to Rüth and Wewers. However, this sometimes does not give sufficient information; for instance, the intersection matrix of the special fiber or the information which points in $X(\mathbb{Q}_\ell)$ reduce to which component cannot always be extracted. For hyperelliptic curves and $\ell \neq 2$, it should also be possible to find the required information using cluster pictures, see [DDMM22] and [BBB+22]. This is used by Betts, Duque-Rosero, Hashimoto and Spelier to give a practical algorithm to compute S_ℓ for hyperelliptic curves such that all components of $\mathcal{X}_\ell$ have genus 0 (i.e. the case of hyperelliptic Mumford curves). When none of these packages give sufficient information, one needs to compute a semistable regular model by hand, which is often not too hard, but tedious. See for instance [BDM+b, Example 5.18].

5.3.2. *Fitting the height pairing.* Step (3) of Algorithm 5.27 consists of writing the height pairing in terms of a given basis $\{\psi_i\}$ of the space $(\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^* \otimes \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*)^*$ of bilinear pairings on $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$. This is analogous to Step (1) of Algorithm 2.37. There, we wrote the Coleman–Gross p-adic height in terms of the basis of bilinear pairings on $J(\mathbb{Q}) \otimes \mathbb{Q}$ given by products of abelian logarithms (i.e. integrals of holomorphic differentials). Since $J(\mathbb{Q}) \otimes \mathbb{Q} \simeq \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ and since, by Remark 3.7, the construction of Coleman–Gross and Nekovář result in the same height, we can also use this approach in the present situation, provided we have an algorithm for the computation of the Coleman–Gross height. This approach is currently restricted in practice to hyperelliptic curves.

In some cases, we may use a more efficient approach, as we now explain. The task is to pick a basis $\{\psi_i\}$ and to evaluate the ψ_i and the height pairing on sufficiently many elements to determine the latter in terms of the former. Since the height pairing is symmetric by construction, we can restrict to symmetric bilinear pairings.

One source of elements of $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^* \otimes \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ comes from representations $A(x)$ for rational points $x \in X(\mathbb{Q})$. The advantage is that we already know how to compute $h_p(A(x))$ for these. So if we

[19] https://github.com/MCLF/mclf

have sufficiently many $x \in X(\mathbb{Q})$ to generate $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^* \otimes \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ using

$$\pi_1(A(x)) \otimes \pi_2(A(x)) = \log([(x) - (b)]) \otimes (E_Z(\log([(x) - (b)])) + c_Z),$$

(see (59) and the discussion preceding it) and if we can compute $h_\ell(A(x))$ for all $\ell \neq p$ (for instance if they all vanish), then we can compute the coefficients of h in terms of the dual basis $\{\psi_i\}$. In practice, we can read off $\pi_i(A(x))$ from our explicit description of the Hodge filtration and Frobenius structure on $A(x)$ determined in §5.2.2 and §5.2.3, respectively. Namely, for $x \in Y(\mathbb{Q})$, we have

$$\pi_1(A(x)) \otimes \pi_2(A(x)) = \alpha_\phi(b,x)^\intercal \cdot \begin{pmatrix} I_g \\ 0_g \end{pmatrix} \otimes \left(\beta_\phi^\intercal(b,x) - \beta_{\mathrm{Fil}}^\intercal(b)\right) \cdot \begin{pmatrix} 0_g \\ I_g \end{pmatrix}. \tag{81}$$

The number of rational points required for this approach can be decreased by working with $\mathrm{End}^0(J)$-equivariant heights, in the sense that $h(P, f(Q)) = h(f(P), Q)$ for all $f \in \mathrm{End}^0(J)$ and $P, Q \in J(\mathbb{Q})$. By [BD21, §4.1], this holds whenever the splitting s of the Hodge filtration on V_{dR} commutes with $\mathrm{End}^0(J)$. In this case, we may determine h in terms of a basis of

$$(\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^* \otimes_{\mathrm{End}^0(J) \otimes \mathbb{Q}_p} \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^*)^*.$$

Example 5.34. If p is ordinary and s is the unit root splitting, then the height is equivariant.

However, even if we use equivariant heights, and if we use $\mathrm{rk}(\mathrm{NS}(J)) - 1$ independent nice correspondences, it is often the case that we do not have enough rational points on X to apply this approach. We currently have no algorithm to compute the height using Nekovář's construction for mixed extensions not of the form $A(x)$, so we have to resort to the above-mentioned method via the Coleman–Gross construction. As discussed, this is only implemented for hyperelliptic curves.

5.4. **QCMod.** Here we explain the QCMod package, which is written in `Magma` and available on Github [BDM+a]. The main function, `QCModAffine`, is in the file `qc_modular.m`.

`QCModAffine` takes as input the following:

- `Q`: a bivariate polynomial with integer coefficients, defining an affine plane curve such that its smooth projective model X and Jacobian $J = \mathrm{Jac}(X)$ satisfy $\mathrm{rk}(J/\mathbb{Q}) = g(X)$ and J has RM over $\mathbb{Q}$. (Note that these conditions are not checked.)
- `p`: a prime of good reduction, satisfying Assumption 1 from Tuitman's algorithm. (The conditions in Assumption 1 are checked.)

There are several optional inputs; here we list a few, such as

- `N`: the p-adic precision used in the computations
- `prec`: the t-adic precision used for power series computations
- `basis0`: a basis of the holomorphic differentials
- `basis1`: a set of g independent meromorphic differentials such that the union of `basis0` and `basis1` generates $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})$

The output is as follows:

- `good_affine_rat_pts_xy`: a list of rational points (x, y) in good residue disks (see Definition 1.48) such that $Q(x, y) = 0$
- `bool`: true if and only if the computation proves that `good_affine_rat_pts_xy` is the complete list of affine rational points in good residue disks
- `bad_affine_rat_pts_xy`: a list of bad rational points (x, y) such that $Q(x, y) = 0$

- `data`: the Coleman data [20] at `p` used in the algorithm. This is a record containing a number of items, such as the matrices W^0, W^∞ and the basis of $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})$.
- `fake_rat_pts`: a list of solutions to the quadratic Chabauty equations which appear to be non-rational. This should be empty if and only if `bool` is true.
- `bad_Qppoints`: a list of $\mathbb{Q}_p$ points representing bad residue disks

The function then has the following main steps:

- Initialization: check and increase precision if needed.
- Symplectic basis: Compute a basis of $H^1_{\mathrm{dR}}(X/\mathbb{Q})$ that is symplectic with respect to the cup product pairing, given a basis of $2g$ differentials, with the first g holomorphic.
- Sort and set known rational points.
- Compute correspondences.
- Compute the Hodge filtration for each correspondence.
- Compute the Frobenius structure for each correspondence.
- Compute heights.
- Expand the quadratic Chabauty function for each correspondence and find its zeros.
- Check for common solutions and compare against the known rational points.

Example 5.35. Note that the function `QCModQuartic` can be called on smooth plane quartics: it wraps `QCModAffine` on two suitable affine patches (satisfying Assumption 1) that it finds covering the curve. For instance, the following code snippet computes the rational points on the split Cartan curve of level 13:

```
load "qc_modular.m";
S0 := CuspidalSubspace(ModularSymbols(169, 2));
S := AtkinLehnerSubspace(S0, 169, 1);
Q := y^3 + y^2*x^2 + y^2*x + y^2 + y*x^3 + y*x - y + x^3 - 3*x^2 + x;
QCModQuartic(Q, S : N := 25);
```

and produces the output

```
Rational points on Xns(13)+
[ (1 : -1 : 1), (2 : -2 : 1), (0 : 0 : 1), (-1/2 : -1 : 1), (-1 : 1 : 0),
  (1 : 0 : 0), (0 : 1 : 0) ]
```

5.5. **An example.** Let N be a positive integer and consider the Atkin–Lehner involution w_N. Then the quotient

$$X_0^+(N) := X_0(N)/\langle w_N \rangle$$

is a nice curve whose non-cuspidal points classify unordered pairs $\{E_1, E_2\}$ of elliptic curves admitting an N-isogeny between them.

In this section we illustrate the quadratic Chabauty method by computing the rational points on $X := X_0^+(167)$. The `Magma` file for this computation can be found here:
https://github.com/steffenmueller/QCMod/blob/main/Examples/qc_X0167plus.m

It was shown by Galbraith [Gal96] that X has genus 2 and that

$$y^2 = x^6 - 4x^5 + 2x^4 - 2x^3 - 3x^2 + 2x - 3$$

[20] This is the Coleman data produced in running Tuitman's algorithm, as described in Balakrishnan–Tuitman's implementation [BTb] of Coleman integration [BT20]; for more details, see the examples file available here: [BTa].

is a model for X. There are four small rational points on X, namely the two points at infinity and $(1, \pm 1)$.

Here is Magma code setting this up:

```
_<x> := PolynomialRing(Rationals());
 f167 := x^6 - 4*x^5 + 2*x^4 - 2*x^3 - 3*x^2 + 2*x - 3;
X := HyperellipticCurve(f167);
```

We then apply a change of variables to move rational points away from infinity:

```
X := Transformation(X, [0,1,1,3]);
```

This is because if there are no rational points at infinity mod p, then we only need quadratic Chabauty on one affine patch; the disks at infinity can be handled via the Mordell–Weil sieve. Our working model is then

$$y^2 = 1881x^6 - 3328x^5 + 2418x^4 - 926x^3 + 197x^2 - 22x + 1. \tag{82}$$

The Jacobian $J = J_0^+(167)$ of X is absolutely simple:

```
J := Jacobian(X);
assert HasAbsolutelyIrreducibleJacobian(X, 1000 : printlevel := 0);
```

This is checked using an implementation of the criterion of Howe and Zhu (see [HZ02, §3]). Then we search for rational points of small height:

```
N := 15;
f := HyperellipticPolynomials(X);
gX := Genus(X);
ptsX := Points(X:Bound:=100);
"Small points: ", ptsX;
```

This yields the points

```
Small points:  {@ (0 : -1 : 1), (0 : 1 : 1), (1 : -1 : 2), (1 : 1 : 2) @}.
```

The Jacobian of this curve has real multiplication, so the Picard number is 2. Using Magma, we compute that the rank of $J(\mathbb{Q})$ is also 2. We can do this in two different ways:

- via a 2-descent on J;
- by computing that the analytic rank of the unique (up to conjugation) newform of level 167 and weight 2 invariant under w_{167} is equal to 1; we may then conclude $\operatorname{rk} J(\mathbb{Q}) = 2$ by the work of Gross–Zagier and Kolyvagin–Logachev. See [DLF21] for details.

The following checks that the rank is two via 2-descent. The function `generators` invokes Magma's `MordellWeilGroupGenus2` to find generators of $J(\mathbb{Q})$.

```
torsion_bas, torsion_orders, bas := generators(J);
assert #bas eq 2; // rank = 2
bas[2] := -bas[2]; // This works better in this particular example.
```

We fix the prime $p = 7$ of good ordinary reduction. Then the logarithm gives an isomorphism $J(\mathbb{Q}) \otimes \mathbb{Q}_7 \to \mathrm{H}^0(X_{\mathbb{Q}_7}, \Omega^1)^*$. According to Theorem 5.2, all requirements for quadratic Chabauty are satisfied, and we may follow Algorithm 5.27 to compute a finite set of 7-adic points containing $X(\mathbb{Q})$. The following code checks that $p = 7$ is a prime such that T_p generates the Hecke algebra. For this computation, we work over the rationals to avoid numerical issues over the p-adics.

```
primes := [7];
exponents := [3];
p := primes[1];
S0 := CuspidalSubspace(ModularSymbols(167, 2));
S := AtkinLehnerSubspace(S0, 167, 1);
assert hecke_operator_generates(S, p);
```

We then compute local heights of representatives for generators of $J(\mathbb{Q}) \otimes \mathbb{Q}$ at p. This is done after computing generators and intersection data for Coleman–Gross heights. We defer a discussion of this to Step (3) of what follows. Below we start by describing what happens internally after

```
good_affine_rat_pts_xy, no_fake_pts, bad_affine_rat_pts_xy, data, fake_rat_pts, bad_Qppoints :=
     QCModAffine(y^2-f, p : printlevel := 1, N := 20, unit_root_splitting := true,
          base_point := base_pt, height_coeffs := height_coeffs, use_log_basis := true);
```

is called.

5.5.1. *Step (1): Expand $h_7(A(x))$.* The most involved step is the computation (and expansion) of the local height $h_7(A(x))$ for a nice correspondence Z via the explicit formula (60). See [BBB+21, Section 6] for a more detailed description of the analogous computation for $X_0^+(67)$. We fix the unit root splitting s of the Hodge filtration on $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_7)$ and the standard cyclotomic idèle class character χ having $\chi_7 = \log_7$, the Iwasawa branch of the 7-adic log.

We first find a symplectic basis $(\omega_0, \dots, \omega_3)$ of $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})$, i.e., one constructed so that the cup product is the standard symplectic form with respect to $(\omega_0, \dots, \omega_3)$. This is done by first computing a basis $\{\omega_0, \omega_1, \eta_2, \eta_3\}$ of $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})$ (originally part of Tuitman's `pcc_q` package, since it is used as input in Tuitman's algorithm), ordered such that ω_0 and ω_1 are holomorphic.

Then one computes the cup product matrix: the matrix whose entries are given by taking cup products of the Tuitman basis. Using the cup product matrix and linear algebra, one can compute the differentials ω_2 and ω_3 such that $\{\omega_0, \omega_1, \omega_2, \omega_3\}$ is symplectic with respect to the cup product pairing. Our subsequent computations will be in terms of this basis.

Via Tuitman's algorithm (Algorithm 1.53), we determine the matrix of Frobenius F on $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_7)$; modulo 7^2, it is given by

$$\begin{pmatrix} 2\cdot 7 & 2\cdot 7 & -3\cdot 7 & -7 \\ -3\cdot 7 & -7 & -7 & -7 \\ 7 & 4 & -13 & 22 \\ 8 & 17 & 24 & 1 \end{pmatrix}.$$

This matrix can be produced by calling `data‘F` and reading off the top left 4×4 block. Note that the rows of this matrix are giving the action of Frobenius on $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_7)$, which is why we use its transpose in the next step.

Eichler–Shimura[21] then allows us to compute the matrix representing the Hecke operator on $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_7)$:

$$T_7 = F^{\intercal} + 7 \cdot (F^{\intercal})^{-1} \begin{pmatrix} 1 & 1 & 0 & -883/2358774 \\ -11 & -6 & 883/2358774 & 0 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 1 & -6 \end{pmatrix}.$$

[21]There are other methods for the computation of Hecke operators, but we have to compute the matrix of Frobenius later on, so we might as well use it here as well.

where F denotes the matrix of 7-adic Frobenius as above. From this we obtain the following matrix representing the endomorphism on $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_7)$ corresponding to a nice correspondence

$$Z = (\mathrm{Tr}(T_7)\cdot I_4 - 4T_7)C^{-1} = \begin{pmatrix} 0 & -1766/1179387 & -14 & -4 \\ 1766/1179387 & 0 & 44 & 14 \\ 14 & -44 & 0 & 0 \\ 4 & -14 & 0 & 0 \end{pmatrix},$$

where $C = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ for $g = 2$ is the matrix of the cup product on our symplectic basis $\{\omega_i\}$. The next step is the computation of the Hodge filtration (see §5.2.2). We use the base point $b = (0, -1)$ and the affine patch Y cut out by our defining equation (82), so we only need a single differential ω_4 of the third kind, having a pole of order 1 at the two points ∞^+, ∞^- at infinity.

These points are defined over $L = \mathbb{Q}(\alpha)$, where α is a root of $x^2 - x - 52$. We apply Algorithm 5.20, which amounts to comparing principal parts of the $g_{\infty^\pm}$ and terms involving $\Omega_{\infty^\pm}$. Since X is hyperelliptic, we have that η and β_{Fil} are trivial by Remark 5.18. We will not use this fact, but instead confirm it using Algorithm 5.20. Expanding and integrating $\vec{\omega}$ around the points at infinity, we find that $d\Omega^{\intercal}_{\infty^\pm} Z\Omega_{\infty^\pm}$ has trivial residue for $i = 1, 2$, so that $\eta = 0$ (see (66)) and $g_{\infty^\pm}$ can simply be taken as the integral of $\vec{\Omega}^{\intercal}_{\infty^\pm} Z d\vec{\Omega}_{\infty^\pm}$. We compute that $g_{\infty^\pm}$ has principal part $\pm 1/19(6656\alpha - 3328)t^{-1}_{\infty^\pm}$ and $g_{\infty^\pm} - \vec{\Omega}^{\intercal}_{\infty^\pm} ZNN^{\intercal}\vec{\Omega}_{\infty^\pm}$ has principal part $\pm 1/57(13312\alpha - 6656)t^{-1}_{\infty^\pm}$ at $\infty^\pm$. Since the principal part of the function x at $\infty^\pm$ is $\pm 1/627(-3328\alpha + 1664)t^{-1}_{\infty^\pm}$ and since our base point has x-coordinate 0, we find that $b_{\mathrm{Fil}} = (0, 0)$ (and hence $\beta_{\mathrm{Fil}} = \vec{0}$) and $\gamma_{\mathrm{Fil}} = -44x$.

We compute the Frobenius structure as in Algorithm 5.26. We have already computed the Frobenius matrix F; the vector $\vec{f}$ of overconvergent functions is a byproduct. The main step is to apply Tuitman's algorithm to the differential

$$\xi := (\phi^* \vec{\omega}^{\intercal}) Z \vec{f} + (F^{\intercal} Z \vec{f})^{\intercal} \vec{\omega}$$

to find $\vec{c}$ and h such that $\vec{c}^{\intercal}\vec{\omega} + dh = \xi$. Modulo 7^2, we have $\vec{c} = (-7, -7, 7, -21)$. We do not write down the expansions of the functions f_i, g_i or h, since these are quite unwieldy.

The next step in Algorithm 5.26 is the computation of the matrix $M(b_0, x_0)$ for x_0 a Teichmüller point in a good disk. We give some details for the residue disk of our base point $b = (0, -1)$; since $b = b_0$ is Teichmüller, we need to compute $M(b, b)$. We have $h(b) = 0$ and $\vec{f}(b) = \vec{0}$, so we only need to compute the $2g$ middle terms of the bottom row of $M(b, b)$. We compute $\vec{g}(b) \equiv (-7, -7, 7, -21) \bmod 7^2$, which implies that $\vec{g}^{\intercal}(F - p)^{-1}(b) \equiv (15, 18, -7, 21) \bmod 7^2$. Since we have $I^-(b_0, b) = 0$, it then suffices to compute the parallel transport matrix $I^+(x(t), b_0)$, where t is a local parameter at b, to find the Frobenius structure $s^\phi(b, x)$ in the disk of b.

Equation (60) then allows us to expand the function $x \mapsto h_7(A(x))$ into a 7-adic power series on those residue disks of $X(\mathbb{Q}_7)$ where our lift of Frobenius is defined. Using the above, we find that in the residue disk of b, we have

$$h_7(A(x)) = t - 5t^2 - 24t^3 + O(t^4, 7^2). \tag{83}$$

5.5.2. *Step (2): Find the possible values of ρ.* This step requires us to find the possible values of $h_\ell(A(x))$ for $\ell \neq 7$ and $x \in X(\mathbb{Q}_\ell)$. Fortunately these are all trivial by Example 5.32.

5.5.3. *Step (3): Determine the height pairing as a bilinear pairing.* Since $X(\mathbb{Q})$ consists of two pairs of points swapped by the hyperelliptic involution, we do not have enough rational points on X to determine the height pairing as a bilinear pairing using only 7-adic heights of the form $h(A(x))$ for $x \in X(\mathbb{Q})$, even

taking $\mathrm{End}^0(J)$-equivariance into account. Instead we determine the height pairing between points in $J(\mathbb{Q})$ via the Coleman–Gross construction.

A basis of $J(\mathbb{Q})$ is given by the points with Mumford representation $P = (x^2 - 7/13x + 1/13, 17/169x - 8/169)$ and $Q = (x^2, 11x - 1)$; this can be computed using the methods[22] of [Sto02, MS16].

For our computations, we use the following divisors

$$D_1 = D_0 - \mathrm{div}_0(x-3), \quad D_1' = \mathrm{div}_0(x-4) - D_0'$$
$$D_2 = 2(0,1) - \mathrm{div}_0(x-4), D_2' = \mathrm{div}_0(x-3) - 2(0,-1),$$

where D_0 (resp. D_0') is the divisor cut out by $x^2 - 7/13x + 1/13$ and $y - (17/169x - 8/169)$ (resp. $y + (17/169x - 8/169)$). Then we can compute the local height pairings $h_v(D_1, D_2)$ and $h_v(D_i, D_i')$ for $i = 1, 2$, noting that their base changes to $X(\mathbb{Q}_7)$ split as sums of $\mathbb{Q}_7$-rational points.

The model (82) is regular outside 2. While the curve X has good reduction at 2, the model (82) does not (the reduction modulo 2 is not reduced), but a regular model can be found easily. Using `Magma`, we find

$$\sum_{\ell \neq p} h_\ell(D_1, D_1') = -\log 97 - \log 181,$$
$$\sum_{\ell \neq p} h_\ell(D_1, D_2) = 2\log 3 + \log 181,$$
$$\sum_{\ell \neq p} h_\ell(D_2, D_2') = -4\log 2 - 2\log 3.$$

In order to compute the local height pairings at 7, we move the unique Weierstrass point in $X(\mathbb{Q}_7)$ to infinity and work with the corresponding odd degree model of X over $\mathbb{Q}_7$ as required by our current `Sage`-implementation. Algorithm 2.28 gives

$$h_7(D_1, D_1') = 3 \cdot 7 + 6 \cdot 7^2 + 7^4 + 6 \cdot 7^5 + 5 \cdot 7^6 + 3 \cdot 7^7 + 3 \cdot 7^8 + 2 \cdot 7^9 + O(7^{10}).$$
$$h_7(D_1, D_2) = 3 \cdot 7 + 6 \cdot 7^3 + 2 \cdot 7^4 + 4 \cdot 7^5 + 7^7 + 5 \cdot 7^8 + 7^9 + O(7^{10}),$$
$$h_7(D_2, D_2') = 3 \cdot 7 + 6 \cdot 7^2 + 3 \cdot 7^3 + 6 \cdot 7^4 + 3 \cdot 7^5 + 2 \cdot 7^6 + 4 \cdot 7^7 + 2 \cdot 7^8 + 3 \cdot 7^9 + O(7^{10}).$$

As described in Step (1) of Algorithm 2.37, we can now express the height h in terms of products of single integrals with coefficients

$$\alpha_{00} = 6 \cdot 7^{-1} + 5 + 2 \cdot 7 + 5 \cdot 7^2 + 2 \cdot 7^3 + 7^6 + 5 \cdot 7^7 + 5 \cdot 7^8 + 5 \cdot 7^9 + O(7^{10})$$
$$\alpha_{01} = 5 \cdot 7^{-1} + 2 + 4 \cdot 7 + 7^2 + 2 \cdot 7^4 + 7^5 + 4 \cdot 7^6 + 6 \cdot 7^7 + 7^9 + O(7^{10})$$
$$\alpha_{11} = 4 \cdot 7^{-1} + 6 + 5 \cdot 7 + 4 \cdot 7^2 + 3 \cdot 7^3 + 2 \cdot 7^4 + 5 \cdot 7^5 + 5 \cdot 7^6 + 2 \cdot 7^9 + O(7^{10}).$$

Hence we obtain our desired function

$$\rho \colon X(\mathbb{Q}_7) \to \mathbb{Q}_7$$

as in Theorem 5.2.

For instance, in the residue disk of b, we find that

$$\rho(x) = h_7(A(x)) - (-10t + 5 \cdot 7^{-1} t^2 - 12t^3) + O(t^4, 7^2)$$

[22] Strictly speaking, a basis of a finite index subgroup is enough, as long as we can show that a given prime does not divide the index.

so (83) implies

$$\rho(x) = 17t + 9 \cdot 7^{-1}t^2 - 12t^3 + O(t^4, 7^2)\,.$$

In this disk, ρ has the zero $t = 0$, recovering the rational point $b = (0, -1)$. It also has a root $t \equiv 9 \bmod 7^2$, which corresponds to the 7-adic point $(14, 6) \bmod 7^2$.

Running through all residue disks, we find that ρ indeed vanishes on the four known rational points; we also see that it has the additional zeros

$$\begin{aligned}
&(2 \cdot 7 + O(7^2), \pm(1 + 6 \cdot 7 + O(7^2))),\\
&(6 + 7 + O(7^2), \pm(3 + O(7^2))),\\
&(6 + 5 \cdot 7 + O(7^2), \pm(3 + 5 \cdot 7 + O(7^2))),\\
&(4 + 3 \cdot 7 + O(7^2), \pm(1 + 6 \cdot 7 + O(7^2))).
\end{aligned}$$

This means that we have computed $X(\mathbb{Q}_7)_U \cap Y(\mathbb{Q}_7) \setminus D_{\mathrm{bad}}$, where D_{bad} is the residue disk of the unique Weierstrass point $(1, 0) \in X(\mathbb{F}_7)$ (where our Frobenius lift is not defined). Note that we have capped the 7-adic precision above for expository purposes. Out implementation starts with a working precision $O(7^{15})$ in this example; according to the precision analysis in [BDM+b, §4], the coefficients of ρ are correct to at least 12 digits and the roots of ρ are correct to at least 10 digits of precision.

Since the Picard number is 2, we cannot show that these do not come from a rational point using an additional nice correspondence, see Remark 5.9. Instead we apply the Mordell–Weil sieve with the auxiliary integer 1045 and the primes $v \in \{3, 5, 19, 29, 31, 67, 263, 281, 283, 769, 1151, 2377, 3847, 4957, 67217\}$.

So now we have shown that $X_0^+(167)$ consists only of the four known rational points – almost. We still have to deal with the disks at infinity and the disk D_{bad}. But since we do not expect any rational points in these disks, we can use the Mordell–Weil sieve to prove this. Since $J(\mathbb{F}_7) \cong \mathbb{Z}/109\,\mathbb{Z}$, we need primes v such that $109 \mid \#J(\mathbb{F}_v)$, which does not happen too often and makes the computation quite involved[23]. But using the auxiliary integer 60, we finally succeed in proving that none of these disks contain a rational point. Comparing with [Gal96, Table 7], we obtain the following:

Theorem 5.36. *There are exactly four rational points on $X_0^+(167)$, and they are all cusps or CM-points.*

5.6. **Some subsequent work on quadratic Chabauty.** During the 2020 Arizona Winter School, we proposed a few projects, which have since been completed. Here we describe two.

5.6.1. *Quadratic Chabauty on the modular curves $X_0^+(N)$.* Galbraith [Gal96, Gal99, Gal02] constructed models for all modular curves $X_0^+(N)$ of genus ≤ 5 (with the exception of $N = 263$) and conjectured that he had found all exceptional points on these curves. Nikola Adžaga, Vishal Arul, Lea Beneish, Mingjie Chen, Shiva Chidambaram, Timo Keller, and Boya Wen [AAB+] studied the case of prime level ℓ in genus 4, 5, and 6, which is the set of

$$\ell \in \{137, 173, 199, 251, 311; 157, 181, 227, 263; 163, 197, 211, 223, 269, 271, 359\}.$$

Prior work in this area include the work of Balakrishnan, Best, Bianchi, Lawrence, Müller, Triantafillou, and Vonk [BBB+21] on the genus 2 prime levels $\ell = 67, 73, 103$, the work of Balakrishnan, Dogra, Müller, Tuitman, and Vonk [BDM+19, BDM+b] on the genus 2 and 3 levels $N = 107, 167, 191; 97, 109, 113, 127, 139, 149, 151, 169, 179, 239$, the work of Balakrishnan, Besser, Bianchi, and Müller [BBBM21] on $N = 91$, and the work of Arul and Müller [AM] on $N = 125$.

[23] We end up using several primes, including two 5-digit primes.

Collectively, these results, plus work of Momose [Mom87], Galbraith [Gal02], and Arai–Momose [AM10] settled the 2002 conjecture of Galbraith: that if $2 \leq g(X_0^+(N)) \leq 5$, then $X_0^+(N)(\mathbb{Q})$ contains exceptional rational points if and only if $N \in \{73, 91, 103, 125, 137, 191, 311\}$.

5.6.2. *Quadratic Chabauty and p-adic L-functions.* The quadratic Chabauty method as we described here requires knowing a number of rational points on the curve or $r = g$ independent points of infinite order on the Jacobian. Both of these conditions are somewhat restrictive, the former for theoretical reasons (the curve very well may not have so many rational points) and the latter for computational reasons (the implementation of Coleman–Gross heights on the Jacobian is currently only for hyperelliptic curves). Fundamentally, both of these conditions are so that one may compute p-adic heights and p-adic logarithms to relate the global p-adic height function in terms of a basis of bilinear forms.

Netan Dogra and Jan Vonk suggested bypassing these restrictions by using p-adic Gross–Zagier theorems to translate p-adic heights and p-adic logarithms into special values of L-functions. Hashimoto [Has] did this for quotients of $X_0(N)$ whose Jacobians are simple quotients of $J_0(N)^{\text{new}}$ over $\mathbb{Q}$.

Acknowledgements. We are very grateful to Vishal Arul, Francesca Bianchi, Stephanie Chan, Netan Dogra, Enis Kaya, and Timo Keller for detailed feedback on an earlier draft of these notes. Additionally, special thanks are due to the participants of the Boston University Fall 2019 course MA 841: Oana (Adascalitei) Pădurariu, Alex Best, María Inés de Frutos Fernández, Stevan Gajović, Sachi Hashimoto, Aashraya Jha, Wanlin Li, Ricky Magner, Angus McAndrew, John Sim, Yifan Wu, and Susan Ye. JB was partially supported by NSF grants DMS-1702196 and DMS-1945452, the Clare Boothe Luce Professorship (Henry Luce Foundation), Simons Foundation grant #550023, and a Sloan Research Fellowship. SM was supported by NWO Grant VI.Vidi.192.106.

Appendix A. Some nonabelian group cohomology

We collect some results on nonabelian group cohomology, closely following Serre [Ser02, §I.5].

Let G be a profinite group. Consider the category of G-*sets*: an object E in this category is a discrete topological space on which G acts continuously, and a morphism between G-sets E_1 and E_2 is a map $f\colon E_1 \to E_2$ that commutes with the action of G. If E is a G-set, $s \in G$ and $x \in E$, the image of x under s will be denoted by ${}^s x$. A G-*group* A is a group in the category of G-sets. This means that A is a G-set, with a group structure that is invariant under G, such that ${}^s(xy) = {}^s x\ {}^s y$ for all $x, y \in A$ (Note that when A is commutative, this gives a G-module.)

If E is a G-set, we let

$$\mathrm{H}^0(G, E) = E^G,$$

the set of elements of E fixed by G. If E is a G-group, then $\mathrm{H}^0(G, E)$ is a group. If A is a G-group, then a 1-cocycle of G in A is a map $s \mapsto a_s$ of G to A that is continuous and satisfies $a_{st} = a_s\ {}^s a_t$ for $s, t \in G$. We denote the set of these cycles by $Z^1(G, A)$. Two cocycles a and a' are cohomologous if there exists $b \in A$ such that $a'_s = b^{-1} a_s\ {}^s b$. This is an equivalence relation $\sim$ on $Z^1(G, A)$, and the quotient set is denoted by

$$\mathrm{H}^1(G, A) = Z^1(G, A)/\sim .$$

We now give another useful interpretation of $\mathrm{H}^1(G, A)$ for a G-group A. We say that A *acts on the left* on a G-set E if it acts on E in the usual way and if ${}^s(a \cdot x) = {}^s a \cdot {}^s x$ for all $a \in A, x \in E$. An action on the right is defined analogously. A G-*equivariant (left*[24]*)* A-*torsor* is a non-empty G-set P on which

[24]Right A-torsors are defined analogously.

A acts on the left, such that for each pair $x, y \in P$, there exists a unique $a \in A$ such that $y = a \cdot x$. We have the following:

Proposition A.1 ([Ser02, Prop. 33])**.** *Let A be a G-group. There is a bijection between the equivalence classes of G-equivariant A-torsors and the set $\mathrm{H}^1(G, A)$.*

Note that while $\mathrm{H}^0(G, A)$ is a group, $\mathrm{H}^1(G, A)$ is merely a *pointed set* when G is non-abelian: it has no group structure, but a distinguished element, given by the class of the unit cocycle. Moreover, the association $A \mapsto \mathrm{H}^i(G, A)$ is functorial for $i = 0, 1$. We can talk about exact sequences of pointed sets (where the image of a map is the inverse image of the neutral element). For instance, we get the following important result:

Proposition A.2 (Six-term exact sequence in non-abelian cohomology [Ser02, Prop. 38])**.** *Let*

$$1 \to A \to B \to C \to 1$$

be a short exact sequence of G-groups. The following sequence of pointed sets:

$$1 \to \mathrm{H}^0(G, A) \to \mathrm{H}^0(G, B) \to \mathrm{H}^0(G, C) \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(G, B) \to \mathrm{H}^1(G, C)$$

is exact.

However, some desirable features are lacking: for instance, injectivity does not follow from having a trivial kernel. More generally, we would like to determine fibers of maps between pointed sets $\mathrm{H}^1(G, A) \to \mathrm{H}^1(G, B)$. Serre's twisting construction, motivated by the theory of fiber bundles, and described below, makes it possible to turn fibers into kernels.

There are analogous constructions when G and A are topological groups, and G acts continuously on A. Considering continuous cocycles and continuous G-equivariant A-torsors yields the continuous cohomology set $\mathrm{H}^1(G, A)$, and the results of [Ser02, §I.5.3] remain valid. Henceforth, we shall assume that we are in this setting, and we shall mostly omit the word "continuous".

A.1. **The twisting construction.** Let G be a topological group, let A be a topological group with a continuous G-action, and let P be a continuous G-equivariant A-torsor. Let F be a G-set on which A acts on the right. We form the *twist* of F by P as follows: consider the equivalence relation that identifies an element (f, p) with $(a \cdot p, fa^{-1})$, for $a \in A$. This relation is compatible with the action of G, and the quotient $F \times_A P$ is a G-set. An element of $F \times_A P$ can be written as $f \cdot p$ for $p \in P, f \in F$, and one has $f(ap) = (fa)p$. Note that for all $p \in P$, the map $f \mapsto f \cdot p$ is a bijection of F onto $F \times_A P$ For this reason, one says that $F \times_A P$ is obtained from F by twisting it using P. This construction gives P the structure of a G-equivariant $F \times_A P$-torsor. We write $A^{(P)} := A \times_A P$, where A acts on itself by conjugation. This construction is easily seen to be functorial in A.

Proposition A.3 ([Ser02, Prop. 35])**.** *Let P be a G-equivariant A-torsor. Then there is a bijection $\mathrm{H}^1(G, A) \to \mathrm{H}^1(G, A^{(P)})$, mapping the class of P in $\mathrm{H}^1(G, A)$ to the neutral element of $\mathrm{H}^1(G, A^{(P)})$.*

So if we have a map $\mathrm{H}^1(G, A) \to \mathrm{H}^1(G, B)$ of pointed sets, coming from a G-group homomorphism $A \to B$, and we want to determine the fiber above the image of some G-equivariant A-torsor P, then we can do this using the induced diagram

$$\begin{array}{ccc} \mathrm{H}^1(G, A) & \longrightarrow & \mathrm{H}^1(G, A^{(P)}) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(G, B) & \longrightarrow & \mathrm{H}^1(G, B^{(P \times_A B)}) \end{array}$$

which commutes due to functoriality of the twisting construction [Ser02, §5.4]. This approach is used in [Ser02, §I.5.5] to determine information about images and fibers of the maps in the six-term exact sequence in Proposition A.2.

Remark A.4. We record some additional useful properties of the twisting construction:

(1) Alternatively, the twisting construction can also be described in terms of cocycles, see [Ser02, §I.5.3] and [Beta, §4.0.1].
(2) If $H^1(G, A)$ and $H^1(G, A^{(P)})$ are representable by schemes, then the twisting bijection in Proposition A.3 is an isomorphism of schemes.
(3) If v is a prime and $U/\mathbb{Q}_v$ is the representation of the absolute Galois group G_p of $\mathbb{Q}_p$ on a finitely generated pro-unipotent group in the sense of [Beta, Section 4], then we can describe $\mathrm{H}^1(G_p, U(Q_v))$ via finite-dimensional G_p-equivariant quotients: Writing $U = \varprojlim U_n$ as an inverse limit of such quotients, we have a natural bijection

$$\mathrm{H}^1(G_p, U(\mathbb{Q}_v)) = \varprojlim \mathrm{H}^1(G_p, U_n(\mathbb{Q}_v)).$$

In particular, this applies to pro-unipotent fundamental groups, such as the ones considered by Kim.

References

[AAB+] N. Adžaga, V. Arul, L. Beneish, M. Chen, Chidambaram S., T. Keller, and B. Wen. Quadratic Chabauty for Atkin-Lehner quotients of modular curves of prime level and genus 4, 5, 6. *Acta. Arith.* to appear. ↑5.6.1.

[AM] V. Arul and J. S. Müller. Rational points on $X_0^+(125)$. *Expositiones Mathematicae (Edixhoven memorial volume)*. to appear. ↑5.6.1.

[AM10] Keisuke Arai and Fumiyuki Momose. Rational points on $X_0^+(37M)$. *J. Number Theory*, 130(10):2272–2282, 2010. ↑5.6.1.

[And03] Yves André. Period mappings and differential equations. *From* $\mathbf{C}$ *to* $\mathbf{C}_p$, *MSJ Memoirs*, 12, 2003. ↑3.1.

[Bal] J. S. Balakrishnan. SageMath code. https://github.com/jbalakrishnan/AWS. ↑1.66, 2.30.

[Bal13] J. S. Balakrishnan. Iterated Coleman integration for hyperelliptic curves. In E. W. Howe and K. S. Kedlaya, editors, *ANTS-X: Proceedings of the Tenth Algorithmic Number Theory Symposium*, volume 1 of *Open Book Series*, pages 41–61. Mathematical Sciences Publishers, 2013. ↑1.5, 1.64, 1.65.

[Bal15] J. S. Balakrishnan. Coleman integration for even-degree models of hyperelliptic curves. *LMS J. Comput. Math.*, 18(1):258–265, 2015. ↑1.5, 1.64, 1.65.

[BB12] J. S. Balakrishnan and A. Besser. Computing local p-adic height pairings on hyperelliptic curves. *IMRN*, 2012(11):2405–2444, 2012. ↑1.41, 2.2.1, 2.2.1, 2.26, 2.28, 2.29.

[BB15] Jennifer S. Balakrishnan and Amnon Besser. Coleman–Gross height pairings and the p-adic sigma function. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 2015(698):89–104, 2015. ↑2.1, 2.32, 2.33.

[BB21] J. S. Balakrishnan and A. Besser. Errata for "Computing local p-adic height pairings on hyperelliptic curves". http://math.bu.edu/people/jbala/cg_heights_errata.pdf, 2021. ↑2.2.1.

[BBB+21] J. S. Balakrishnan, A.J. Best, F. Bianchi, B. Lawrence, J. S. Müller, N. Triantafillou, and J. Vonk. Two recent p-adic approaches towards the (effective) Mordell conjecture. In *Regulators IV: An international conference on arithmetic L-functions and differential geometric methods*, volume 338 of *Progr. Math.*, pages 31–74. Birkhäuser Boston, Boston, MA, 2021. ↑1.1, 2.3.2, 4.22, 5.5.1, 5.6.1.

[BBB+22] Alex J. Best, L. Alexander Betts, Matthew Bisatt, Raymond van Bommel, Vladimir Dokchitser, Omri Faraggi, Sabrina Kunzweiler, Céline Maistret, Adam Morgan, Simone Muselli, and Sarah Nowell. A user's guide to the local arithmetic of hyperelliptic curves. *Bull. London Math. Soc.*, To appear, https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12604, 2022. ↑2.39, 5.3.1.

[BBBM21] J. S. Balakrishnan, A. Besser, F. Bianchi, and J. S. Müller. Explicit quadratic Chabauty over number fields. *Israel J. Math*, 243:185–232, 2021. ↑2.2, 2.3, 5.6.1.

[BBK10] J. S. Balakrishnan, R. W. Bradshaw, and K. Kedlaya. Explicit Coleman integration for hyperelliptic curves. In *Algorithmic number theory*, volume 6197 of *Lecture Notes in Comput. Sci.*, pages 16–31. Springer, Berlin, 2010. ↑1.3, 1.3, 1.37, 1.40, 1.41.

[BBM16] Jennifer S. Balakrishnan, Amnon Besser, and J. Steffen Müller. Quadratic Chabauty: p-adic heights and integral points on hyperelliptic curves. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 2016(720):51–79, 2016. ↑2.33, 2.34, 2.35, 2.3.1.

[BBM17] Jennifer S. Balakrishnan, Amnon Besser, and J. Steffen Müller. Computing integral points on hyperelliptic curves using quadratic Chabauty. *Math. Comp.*, 86:1403–1434, 2017. ↑2.30, 2.3, 2.3, 2.3.2, 2.40, 12.

[BC94] Francesco Baldassarri and Bruno Chiarellotto. Algebraic versus rigid cohomology with logarithmic coefficients. In *Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991)*, volume 15 of *Perspect. Math.*, pages 11–50. Academic Press, San Diego, CA, 1994. ↑1.3, 3.1.

[BC09] O. Brinon and B. Conrad. CMI summer school notes on p-adic Hodge theory. 2009. ↑3.1, 3.3.

[BC22] L. Alexander Betts and D. (with an appendix by M. Leonhardt) Corwin. Towards uniform Chabauty–Kim. 2022. https://arxiv.org/pdf/2206.11085.pdf. ↑5.10.

[BCP97] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system I: The user language. *J. Symb. Comp*, 24(3-4):235–265, 1997. ↑1.

[BD18] Jennifer S. Balakrishnan and Netan Dogra. Quadratic Chabauty and rational points I: p-adic heights. *Duke Math. J.*, 167(11):1981–2038, 2018. ↑3.2, 3.2, 4, 4.1, 4.4, 4.12, 4.5, 4.6, 5.1, 5.2, 5.7, 5.9, 5.1, 5.1, 5.1.

[BD19a] Jennifer S. Balakrishnan and Netan Dogra. An effective Chabauty-Kim theorem. *Compos. Math.*, 155(6):1057–1075, 2019. ↑5.10.

[BD19b] L. Alexander Betts and Netan Dogra. The local theory of unipotent Kummer maps and refined Selmer schemes. Sep 2019. https://arxiv.org/abs/1909.05734. ↑5.3.1.

[BD21] J. S. Balakrishnan and N. Dogra. Quadratic Chabauty and rational points II: Generalised height functions on Selmer varieties. *Int. Math. Res. Not. IMRN*, (15):11923–12008, 2021. ↑3.2, 4.2, 4.6, 5, 5.3, 5.1, 5.1, 5.1, 5.2, 5.18, 5.3.2.

[BDCKW18] J. S. Balakrishnan, I. Dan-Cohen, M. Kim, and S. Wewers. A non-abelian conjecture of Tate-Shafarevich type for hyperbolic curves. *Math. Ann.*, 372(1-2):369–428, 2018. ↑4.6, 5.1.

[BDM+a] J.S. Balakrishnan, N. Dogra, J.S. Müller, J. Tuitman, and J. Vonk. Magma code. https://github.com/steffenmueller/QCMod. ↑2.30, 5.4.

[BDM+b] J. S. Balakrishnan, N. Dogra, J. S. Müller, J. Tuitman, and J. Vonk. Quadratic Chabauty for modular curves: Algorithms and examples. https://arxiv.org/abs/2101.01862, to appear, Comp. Math. ↑5.3, 5.3, 5.3.1, 5.32, 5.3.1, 5.33, 5.3.1, 5.5.3, 5.6.1.

[BDM+19] J.S. Balakrishnan, N. Dogra, J.S. Müller, J. Tuitman, and J. Vonk. Explicit Chabauty-Kim for the split Cartan modular curve of level 13. *Ann. of Math. (2)*, 189(3), 2019. ↑3.2, 3.3, 3.3.1, 4.2, 4.4, 4.20, 5.1, 5.2, 5.16, 5.2, 5.2, 5.2, 5.2.2, 5.2.2, 5.19, 5.2.3, 5.2.3, 5.2.3, 5.3, 5.28, 5.3.1, 5.6.1.

[Bel09] J. Bellaïche. CMI summer school notes on an introduction to the conjecture of Bloch-Kato. 2009. ↑3.1, 3.3, 3.4.

[Ber75] Daniel Bertrand. Valeurs algébriques de fonctions méromorphes. In *Séminaire Delange-Pisot-Poitou, 15e année (1973/74), Théorie des nombres, Fasc. 1, Exp. No. 21*, page 6. 1975. ↑2.1.

[Ber81] Dominique Bernardi. Hauteur p-adique sur les courbes elliptiques. In *Seminar on Number Theory, Paris 1979–80*, volume 12 of *Progr. Math.*, pages 1–14. Birkhäuser, Boston, Mass., 1981. ↑2, 2.4.

[Ber96] P. Berthelot. Cohomologie rigide et cohomologie rigide a supports propres. *Preprint*, 1996. ↑5.2.3.

[Ber97] Pierre Berthelot. Finitude et pureté cohomologique en cohomologie rigide. *Invent. Math.*, 128(2):329–377, 1997. With an appendix in English by Aise Johan de Jong. ↑1.3.

[Ber04] Laurent Berger. An introduction to the theory of p-adic representations. *Geometric aspects of Dwork theory*, 1:255–292, 2004. ↑3.1.

[Ber07] Vladimir G. Berkovich. *Integration of one-forms on p-adic analytic spaces*, volume 162 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2007. ↑1.11.

[Bes02] Amnon Besser. Coleman integration using the Tannakian formalism. *Math. Ann.*, 322(1):19–48, 2002. ↑1.11, 1.5, 2.1, 5.2.3.

[Bes04] Amnon Besser. The p-adic height pairings of Coleman-Gross and of Nekovář. In *Number theory*, volume 36 of *CRM Proc. Lecture Notes*, pages 13–25. Amer. Math. Soc., Providence, RI, 2004. ↑3.7.

[Bes05] Amnon Besser. p-adic Arakelov theory. *J. Number Theory*, 111(2):318–371, 2005. ↑2.2.1, 4.13.

[Bes12] Amnon Besser. Heidelberg lectures on Coleman integration. In J. Stix, editor, *The arithmetic of fundamental groups—PIA 2010*, volume 2 of *Contrib. Math. Comput. Sci.*, pages 3–52. Springer, Heidelberg, 2012. ↑1.11, 1.6, 2.1, 5.2.3, 5.2.3, 5.23.

[Bes19] Alex J. Best. Explicit Coleman integration in larger characteristic. In *Proceedings of the Thirteenth Algorithmic Number Theory Symposium*, volume 2 of *Open Book Ser.*, pages 85–102. Math. Sci. Publ., Berkeley, CA, 2019. ↑1.42.

[Bes21] Alex J. Best. Square root time Coleman integration on superelliptic curves. In *Arithmetic geometry, number theory, and computation*, Simons Symp., pages 105–129. Springer, Cham, [2021] ©2021. ↑1.42.

[Bes22] Amnon Besser. p-adic heights and Vologodsky integration. *J. Number Theory*, 239:273–297, 2022. ↑2.32.

[Beta] L. Alexander Betts. The motivic anabelian geometry of local heights on abelian varieties. Mem. Amer. Math. Soc. (to appear). ↑4.22, 1, 3.

[Betb] L. Alexander Betts. Weight filtrations on Selmer schemes and the effective Chabauty–Kim method. *Comp. Math. (to appear)*. ↑5.10.

[BGJGP05] Matthew H Baker, Enrique González-Jiménez, Josep González, and Bjorn Poonen. Finiteness results for modular curves of genus at least 2. *American Journal of Mathematics*, 127(6):1325–1387, 2005. ↑2.30.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry. ↑1.2.

[Bia19] Francesca Bianchi. Topics in the theory of p-adic heights on elliptic curves. *Oxford DPhil thesis*, 2019. ↑2.13.

[Bia20] Francesca Bianchi. Quadratic Chabauty for (bi)elliptic curves and Kim's conjecture. *Algebra Number Theory*, 14(9):2369–2416, 2020. ↑4.1, 4.7.

[BK90] Spencer Bloch and Kazuya Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990. ↑3.12, 4.2, 4.6.

[BKK11] Jennifer S. Balakrishnan, Kiran S. Kedlaya, and Minhyong Kim. Appendix and erratum to "Massey products for elliptic curves of rank 1". *J. Amer. Math. Soc.*, 24(1):281–291, 2011. ↑1.68.

[Bla18] C. Blakestad. *On Generalizations of p-Adic Weierstrass Sigma and Zeta Functions*. PhD thesis, University of Colorado, 2018. ↑2.14.

[BMS16] Jennifer S. Balakrishnan, J. Steffen Müller, and William A. Stein. A p-adic analogue of the conjecture of Birch and Swinnerton-Dyer for modular abelian varieties. *Math. Comp.*, 85(298):983–1016, 2016. ↑2.24.

[BMS21] Amnon Besser, J. Steffen Müller, and Padmavathi Srinivasan. p-adic adelic metrics and quadratic Chabauty I, 2021. https://arxiv.org/pdf/2112.03873.pdf. ↑1, 4.13, 5.8.

[Bom90] Enrico Bombieri. The Mordell conjecture revisited. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 17(4):615–640, 1990. ↑1.1.

[BS10] Nils Bruin and Michael Stoll. The Mordell-Weil sieve: proving non-existence of rational points on curves. *LMS J. Comput. Math.*, 13:272–306, 2010. ↑6, 2.3.2.

[BTa] J.S. Balakrishnan and J. Tuitman. Examples, Coleman Magma repository. https://github.com/jtuitman/Coleman/blob/master/examples.pdf. ↑20.

[BTb] J. S. Balakrishnan and J. Tuitman. Magma code. https://github.com/jtuitman/Coleman. ↑1.23, 1.57, 1.58, 1.59, 20.

[BT20] J. S. Balakrishnan and J. Tuitman. Explicit Coleman integration for curves. *Math. Comp.*, 89(326):2965–2984, 2020. ↑1.4, 1.54, 1.56, 1.57, 20.

[BZ] Amnon Besser and Sarah Livia Zerbes. Vologodsky integration on curves with semi-stable reduction. Israel J. Math, to appear. ↑1.16.

[CdS88] Robert Coleman and Ehud de Shalit. p-adic regulators on curves and special values of p-adic L-functions. *Invent. Math.*, 93(2):239–266, 1988. ↑1.5.

[CDV06] W. Castryck, J. Denef, and F. Vercauteren. Computing zeta functions of nondegenerate curves. *IMRP Int. Math. Res. Pap.*, pages Art. ID 72017, 57, 2006. ↑1.4.

[CG89] Robert F. Coleman and Benedict H. Gross. p-adic heights on curves. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 73–81. Academic Press, Boston, MA, 1989. ↑2, 2.17, 2.18, 2.2.1, 2.2.1, 2.3.1.

[Cha41] C. Chabauty. Sur les points rationels des courbes algébriques de genre supérieur à l'unité. *C.R. Acad. Sci.*, 212:882–884, 1941. ↑1.1.

[Cha16] S. Chan. Topics in the theory of zeta functions of curves. *Oxford MMath thesis*, 2016. https://www.ucl.ac.uk/~ucahytc/chan_dissertation.pdf. ↑1.33, 1.35, 1.36.

[Che71] Kuo-tsai Chen. Algebras of iterated path integrals and fundamental groups. *Trans. Amer. Math. Soc.*, 156:359–379, 1971. ↑1.5.

[CI99] Robert Coleman and Adrian Iovita. The Frobenius and monodromy operators for curves and abelian varieties. *Duke Math. J.*, 97(1):171–215, 1999. ↑5.3.1.

[CK10] John Coates and Minhyong Kim. Selmer varieties for curves with CM Jacobians. *Kyoto J. Math.*, 50(4):827–852, 2010. ↑4.10, 5.1.

[CLS99] Bruno Chiarellotto and Bernard Le Stum. F-isocristaux unipotents. *Compositio Math.*, 116(1):81–110, 1999. ↑5.2.3, 5.25.

[CMSV19] Edgar Costa, Nicolas Mascot, Jeroen Sijsling, and John Voight. Rigorous computation of the endomorphism ring of a Jacobian. *Math. Comp.*, 88(317):1303–1339, 2019. ↑5.7.

[Col82] R. F. Coleman. Dilogarithms, regulators and p-adic L-functions. *Invent. Math.*, 69(2):171–208, 1982. ↑1.9, 1.5.

[Col85a] Robert F. Coleman. Effective Chabauty. *Duke Mathematical Journal*, 52(3):765–770, 1985. ↑1.4.

[Col85b] Robert F. Coleman. Torsion points on curves and p-adic abelian integrals. *Ann. of Math. (2)*, 121(1):111–168, 1985. ↑1.1, 1.9, 1.26.

[Col91] Robert F. Coleman. The universal vectorial bi-extension and p-adic heights. *Invent. Math.*, 103(3):631–650, 1991. ↑2.24.

[Col98] Pierre Colmez. Intégration sur les variétés p-adiques. *Astérisque*, (248):viii+155, 1998. ↑1.11.

[Cor19] David Corwin. From Chabauty's method to Kim's non-abelian Chabauty's method. 2019. https://math.berkeley.edu/~dcorwin/files/ChabautytoKim.pdf. ↑3.12, 4.1.

[DDMM22] T. Dokchitser, V. Dokchitser, C. Maistret, and A. Morgan. Arithmetic of hyperelliptic curves over local fields. *Math. Ann.*, To appear, https://link.springer.com/article/10.1007/s00208-021-02319-y, 2022. ↑2.39, 5.3.1.

[Del89] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over* **Q** *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989. ↑1.6.

[DLF21] Netan Dogra and Samuel Le Fourn. Quadratic Chabauty for modular curves and modular forms of rank one. *Math. Ann.*, 380(1-2):393–448, 2021. ↑4.21, 4.6, 5.10, 5.5.

[DM82] P. Deligne and J. S. Milne. *Tannakian Categories*, pages 101–228. Springer Berlin Heidelberg, Berlin, Heidelberg, 1982. ↑5.2.

[Dok21] Tim Dokchitser. Models of curves over discrete valuation rings. *Duke Math. J.*, 170(11):2519–2574, 2021. ↑2.3.1.

[DRHS22] Juanita Duque-Rosero, Sachi Hashimoto, and Pim Spelier. Geometric quadratic Chabauty and p-adic heights. 2022. https://arxiv.org/pdf/2207.10389.pdf. ↑5.7.

[DRS12] H. Darmon, V. Rotger, and I. Sols. Iterated integrals, diagonal cycles, and rational points on elliptic curves. *Publ. Math. de Besançon*, 2:19–46, 2012. ↑5.1.

[DV06a] Jan Denef and Frederik Vercauteren. Counting points on C_{ab} curves using Monsky-Washnitzer cohomology. *Finite Fields Appl.*, 12(1):78–102, 2006. ↑1.4.

[DV06b] Jan Denef and Frederik Vercauteren. An extension of Kedlaya's algorithm to hyperelliptic curves in characteristic 2. *J. Cryptology*, 19(1):1–25, 2006. ↑1.4.

[Edi] B. Edixhoven. Point counting after Kedlaya, EIDMA-Stieltjes graduate course, Leiden, September 22-26,2003. http://www.math.leidenuniv.nl/~edix/oww/mathofcrypt/carls_edixhoven/kedlaya.pdf. ↑1.2, 1.31.

[EH22] Jordan S. Ellenberg and Daniel Rayor Hast. Rational points on solvable curves over $\mathbb{Q}$ via non-abelian Chabauty. *Int. Math. Res. Not. IMRN*, (19):14770–14796, 2022. ↑4.11.

[EL23] Bas Edixhoven and Guido Lido. Geometric quadratic Chabauty. *J. Inst. Math. Jussieu*, 22(1):279—-333, 2023. ↑1, 4.13, 4.22.

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983. ↑1.2.

[Fal89] Gerd Faltings. Crystalline cohomology and p-adic Galois-representations. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 25–80. Johns Hopkins Univ. Press, Baltimore, MD, 1989. ↑3.1, 3.2.

[FvdP04] Jean Fresnel and Marius van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004. ↑1.1, 1.2.

[Gal96] S. D. Galbraith. Equations for modular curves. *Oxford DPhil thesis*, 1996. ↑5.5, 5.5.3, 5.6.1.

[Gal99] Steven D. Galbraith. Rational points on $X_0^+(p)$. *Experiment. Math.*, 8(4):311–318, 1999. ↑5.6.1.

[Gal02] Steven D. Galbraith. Rational points on $X_0^+(N)$ and quadratic $\mathbb{Q}$-curves. *J. Théor. Nombres Bordeaux*, 14(1):205–219, 2002. ↑5.6.1.

[GG01] Pierrick Gaudry and Nicolas Gürel. An extension of Kedlaya's point-counting algorithm to superelliptic curves. In *Advances in cryptology—ASIACRYPT 2001 (Gold Coast)*, volume 2248 of *Lecture Notes in Comput. Sci.*, pages 480–494. Springer, Berlin, 2001. ↑1.4.

[Gro86] Benedict H Gross. Local heights on curves. In *Arithmetic geometry*, pages 327–339. Springer, 1986. ↑2.2.1.

[Had11] M. Hadian. Motivic fundamental groups and integral points. *Duke Math. J.*, 160(3):503–565, 2011. ↑5.2.

[Har07] D. Harvey. Kedlaya's algorithm in larger characteristic. *Int Math Res Notices*, 2007(rnm095):rnm095–29, 2007. ↑1.42.

[Har08] D. Harvey. Efficient computation of p-adic heights. *LMS J. Comput. Math.*, 11:40–59, 2008. ↑2.10, 2.11.

[Har12] M. C. Harrison. An extension of Kedlaya's algorithm for hyperelliptic curves. *J. Symb. Comp.*, 47(1):89 – 101, 2012. ↑1.4.

[Has] Sachi Hashimoto. Quadratic Chabauty and p-adic Gross–Zagier. *Trans. AMS*. to appear. ↑5.6.2.

[HM] S. Hashimoto and T. Morrison. Magma code. https://github.com/travismo/Coleman. ↑1.59.

[HM19] Yoshinosuke Hirakawa and Hideki Matsumura. A unique pair of triangles. *Journal of Number Theory*, 194:297–302, 2019. ↑1.1.

[Hol12] David Holmes. Computing Néron-Tate heights of points on hyperelliptic Jacobians. *J. Number Theory*, 132(6):1295–1305, 2012. ↑2.3.1.

[HZ02] E W. Howe and H. J. Zhu. On the existence of absolutely simple abelian varieties of a given dimension over an arbitrary field. *J. Number Theory*, 92(1):139–163, 2002. ↑5.5.

[IW03] Adrian Iovita and Annette Werner. p-adic height pairings on abelian varieties with semistable ordinary reduction. *J. Reine Angew. Math.*, 564:181–203, 2003. ↑2.

[Kat73] N. Katz. p-Adic properties of modular schemes and modular forms. In P. Deligne and W. Kuyk, editors, *Modular forms in one variable III*, volume 350 of *LNM*, pages 69–190. Springer-Verlag, 1973. ↑2.1.

[Ked01] Kiran S. Kedlaya. Counting points on hyperelliptic curves using Monsky-Washnitzer cohomology. *J. Ramanujan Math. Soc.*, 16(4):323–338, 2001. ↑3, 1.3, 1.29, 1.30, 1.31.

[Ked03] Kiran S. Kedlaya. Errata for: "Counting points on hyperelliptic curves using Monsky-Washnitzer cohomology" [J. Ramanujan Math. Soc. **16** (2001), no. 4, 323–338; mr1877805]. volume 18, pages 417–418. 2003. Dedicated to Professor K. S. Padmanabhan. ↑3, 1.32.

[Kim05] Minhyong Kim. The motivic fundamental group of $\mathbf{P}^1 - \{0, 1, \infty\}$ and the theorem of Siegel. *Inventiones mathematicae*, 161(3):629–656, 2005. ↑4.1, 4.3, 4.3, 4.5.

[Kim09] M. Kim. The unipotent Albanese map and Selmer varieties for curves. *Publ. RIMS*, 45:89–133, 2009. ↑4.1, 4.1, 4.5, 4.1, 4.1, 4.8, 4.9, 4.5, 4.5, 5.1, 5, 5.2, 5.2.2.

[Kim10] M. Kim. Massey products for elliptic curves of rank 1. *J. Amer. Math. Soc.*, 23(3):725–747, 2010. ↑1.68.

[Kim12] Minhyong Kim. Tangential localization for Selmer varieties. *Duke Math. J.*, 161(2):173–199, 2012. ↑4.1.

[KK22] Eric Katz and Enis Kaya. p-adic integration on bad reduction hyperelliptic curves. *Int. Math. Res. Not. IMRN*, (8):6038–6106, 2022. ↑1.41.

[KL] M. Kim and M. Lüdtke. Foundations of the nonabelian method of Chabauty. 2020 Arizona Winter School Lectures. ↑5.2, 5.2.

[KRZB16] Eric Katz, Joseph Rabinoff, and David Zureick-Brown. Uniform bounds for the number of rational points on curves of small Mordell–Weil rank. *Duke Mathematical Journal*, 165(16):3189–3240, 2016. ↑1.16, 1.22.

[KT08] M. Kim and A. Tamagawa. The l-component of the unipotent Albanese map. *Math. Ann.*, 340(1):223–235, 2008. ↑4.1, 4.3.

[KZB13] Eric Katz and David Zureick-Brown. The Chabauty-Coleman bound at a prime of bad reduction and Clifford bounds for geometric rank functions. *Compos. Math.*, 149(11):1818–1838, 2013. ↑3.

[Lan88] Serge Lang. *Introduction to Arakelov theory*. Springer-Verlag, New York, 1988. ↑2.3.1.

[LMF20a] The LMFDB Collaboration. The L-functions and modular forms database, home page of the genus 2 curve `8832.a.17664.1`. https://www.lmfdb.org/Genus2Curve/Q/8832/a/17664/1, 2020. [Online; accessed 7 February 2020]. ↑1.66.

[LMF20b] The LMFDB Collaboration. The L-functions and modular forms database, home page of the genus 2 curve `971.a.971.1`. https://www.lmfdb.org/Genus2Curve/Q/971/a/971/1, 2020. [Online; accessed 30 January 2020]. ↑1.23.

[LT02] Dino Lorenzini and Thomas J. Tucker. Thue equations and the method of Chabauty-Coleman. *Invent. Math.*, 148(1):47–77, 2002. ↑1.

[LV20] Brian Lawrence and Akshay Venkatesh. Diophantine problems and p-adic period mappings. *Invent. Math.*, 221(3):893–999, 2020. ↑1.1.

[Mil80] J. Milne. *Étale cohomology*. Princeton University Press, 1980. ↑4.4.

[Min10] Moritz Minzlaff. Computing zeta functions of superelliptic curves in larger characteristic. *Mathematics in Computer Science*, 3(2):209–224, 2010. ↑1.42.

[Mom87] Fumiyuki Momose. Rational points on the modular curves $X_0^+(N)$. *J. Math. Soc. Japan*, 39(2):269–286, 1987. ↑5.6.1.

[MP12] William McCallum and Bjorn Poonen. The method of Chabauty and Coleman. In *Explicit methods in number theory*, volume 36 of *Panor. Synthèses*, pages 99–117. Soc. Math. France, Paris, 2012. ↑1.1, 1.13.

[MS16] Jan Steffen Müller and Michael Stoll. Canonical heights on genus-2 Jacobians. *Algebra Number Theory*, 10(10):2153–2234, 2016. ↑5.5.3.

[MST06] Barry Mazur, William Stein, and John Tate. Computation of p-adic heights and log convergence. *Doc. Math*, pages 577–614, 2006. ↑2.1, 2.11.

[MT83] B. Mazur and J. Tate. Canonical height pairings via biextensions. In *Arithmetic and geometry, Vol. I*, volume 35 of *Progr. Math.*, pages 195–237. Birkhäuser Boston, Boston, MA, 1983. ↑2, 2.24.

[MT91] B. Mazur and J. Tate. The p-adic sigma function. *Duke Math. J.*, 62(3):663–688, 1991. ↑2.1, 2.3.

[MTT86] B. Mazur, J. Tate, and J. Teitelbaum. On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent. Math.*, 84(1):1–48, 1986. ↑2.6, 2.11.

[Mül14] J. Steffen Müller. Computing canonical heights using arithmetic intersection theory. *Mathematics of Computation*, 83(285):311–336, 2014. ↑2.3.1, 2.40.

[Nek93] J. Nekovar. On p-adic height pairings. In *Séminaire de Théorie des Nombres, Paris 1990-1991*, pages 127–202. Birkhäuser, 1993. ↑2, 3, 3.1, 3.2, 14, 3.2, 3.6, 3.7, 3.3.1, 5.7.

[Nér76] André Néron. Hauteurs et fonctions thêta. *Rend. Sem. Mat. Fis. Milano*, 46:111–135 (1978), 1976. ↑2.

[Ols11] M. Olsson. Towards non-abelian p-adic Hodge theory in the good reduction case. *Memoirs of the AMS*, (990), 2011. ↑3.1, 4.1, 4.1, 5.2.

[Poo06] Bjorn Poonen. Heuristics for the Brauer-Manin obstruction for curves. *Experiment. Math.*, 15(4):415–420, 2006. ↑2.3.2, 4.1.

[PR83] Bernadette Perrin-Riou. Descente infinie et hauteur p-adique sur les courbes elliptiques à multiplication complexe. *Invent. Math.*, 70(3):369–398, 1982/83. ↑2.

[PSS07] Bjorn Poonen, Edward F. Schaefer, and Michael Stoll. Twists of $X(7)$ and primitive solutions to $x^2+y^3 = z^7$. *Duke Math. J.*, 137(1):103–158, 2007. ↑2.3.2.

[Qui69] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969. ↑5.1.

[Sch82] Peter Schneider. p-adic height pairings. I. *Invent. Math.*, 69(3):401–409, 1982. ↑2, 2.12.

[Sch94] A. J. Scholl. Height pairings and special values of L-functions. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 571–598. Amer. Math. Soc., Providence, RI, 1994. ↑3.1.

[Sch98] Peter Schneider. Basic notions of rigid analytic geometry. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 369–378. Cambridge Univ. Press, Cambridge, 1998. ↑1.1.

[Sch99] Victor Scharaschkin. *Local-global problems and the Brauer-Manin obstruction*. ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–University of Michigan. ↑2.3.2.

[Ser02] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, English edition, 2002. ↑A, A.1, A.2, A, A.3, A.1, 1.

[Sik17] S. Siksek. Quadratic Chabauty for modular curves. *preprint*, 2017. ↑4.

[Smi05] B. Smith. *Explicit endomorphisms and correspondences*. PhD thesis, University of Sydney, 2005. ↑4.4.

[Sto] Michael Stoll. Arithmetic of hyperelliptic curves. ↑1.1.

[Sto99] Michael Stoll. On the height constant for curves of genus two. *Acta Arith.*, 90(2):183–201, 1999. ↑2.40.

[Sto01] Michael Stoll. Implementing 2-descent for Jacobians of hyperelliptic curves. *Acta Arith.*, 98(3):245–277, 2001. ↑2.40.

[Sto02] Michael Stoll. On the height constant for curves of genus two. II. *Acta Arith.*, 104(2):165–182, 2002. ↑5.5.3.

[Sto06] Michael Stoll. Independence of rational points on twists of a given curve. *Compos. Math.*, 142(5):1201–1214, 2006. ↑2.

[Sto19] Michael Stoll. Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank. *J. Eur. Math. Soc. (JEMS)*, 21(3):923–956, 2019. ↑1.16, 1.21.

[SW13] William Stein and Christian Wuthrich. Algorithms for the arithmetic of elliptic curves using Iwasawa theory. *Mathematics of Computation*, 82(283):1757–1792, 2013. ↑2.7, 2.8.

[The19] The LMFDB Collaboration. The L-functions and Modular Forms Database. http://www.lmfdb.org, 2019. [Online; accessed 30 January 2020]. ↑4.7.

[The20] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.0)*, 2020. `https://www.sagemath.org`. ↑1.

[Tui16] Jan Tuitman. Counting points on curves using a map to $\mathbf{P}^1$. *Math. Comp.*, 85(298):961–981, 2016. ↑1.4, 1.4, 1.53, 1.4.

[Tui17] Jan Tuitman. Counting points on curves using a map to $\mathbf{P}^1$, II. *Finite Fields Appl.*, 45:301–322, 2017. ↑1.4, 1.46, 1, 1.4, 1.53, 1.4.

[VBHM20] Raymond Van Bommel, David Holmes, and J. Steffen Müller. Explicit arithmetic intersection theory and computation of Néron-Tate heights. *Mathematics of Computation*, 89(321):395–410, 2020. ↑2.3.1, 2.40.

[Voj91] Paul Vojta. Siegel's theorem in the compact case. *Ann. of Math. (2)*, 133(3):509–548, 1991. ↑1.1.

[Vol03] Vadim Vologodsky. Hodge structure on the fundamental group and its application to p-adic integration. *Mosc. Math. J.*, 3(1):205–247, 260, 2003. ↑1.11.

[Wer98] Annette Werner. Local heights on abelian varieties and rigid analytic uniformization. *Doc. Math.*, 3:301–319, 1998. ↑2.24.

[Wut04] Christian Wuthrich. On p-adic heights in families of elliptic curves. *J. London Math. Soc. (2)*, 70(1):23–40, 2004. ↑2.13.

[Xue09] Hui Xue. Minimal resolution of Atkin-Lehner quotients of $X_0(N)$. *J. Number Theory*, 129(9):2072–2092, 2009. ↑5.32.

[Zar90] Yuri G. Zarhin. p-adic heights on abelian varieties. In *Séminaire de Théorie des Nombres, Paris 1987–88*, volume 81 of *Progr. Math.*, pages 317–341. Birkhäuser Boston, Boston, MA, 1990. ↑2.

[Zar96] Yu. G. Zarhin. p-adic abelian integrals and commutative Lie groups. volume 81, pages 2744–2750. 1996. Algebraic geometry, 4. ↑1.11.

[Zha93] Shouwu Zhang. Admissible pairing on a curve. *Invent. Math.*, 112(1):171–193, 1993. ↑2.

[Zha95] Shouwu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995. ↑1.

J. S. Balakrishnan, Department of Mathematics and Statistics, Boston University, 665 Commonwealth Avenue, Boston, MA 02215, USA

Email address: `jbala@bu.edu`

J. S. Müller, Bernoulli Institute, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands

Email address: `steffen.muller@rug.nl`