Erratum to *A nonabelian conjecture of Tate-Shafarevich type...* [1]

We are grateful to Francesca Bianchi and L. Alexander Betts for pointing out the following errors.

The theorem of paragraph 1.12 is incorrect as stated — instead of an equality in the first displayed equation we only have an inclusion

$$\mathcal{X}(\mathbb{Z}_p)_2 \subset \bigcup_{w \in W} \Psi(w). \tag{*}$$

Correspondingly, an error in the proof of the theorem occurs in the third displayed equation on page 413: the first isomorphism should be replaced by an inclusion as follows:

$$\bigoplus_{v \in S} j_v(\mathcal{X}(\mathbb{Z}_p)) \subset \bigoplus_{v \in S} W_v.$$

To trace the error back, let F_v be unramified over $\mathbb{Q}_l$ ($l \neq p$), let E_v be an elliptic curve over F_v with semistable reduction and with $\mathcal{O}_v$-model $\mathcal{E}$, and let $\mathcal{X} = \mathcal{E} \setminus \{0\}$. In Corollary 4.6 we show that the possible values of our function

$$\phi_v : (\mathcal{E} \setminus \{0\})(\mathcal{O}_v) \to \mathbb{Q}_p$$

are *among* the values

$$-(n(N_v - n)/2N_v) \log l; \quad 0 \leq n < N_v. \tag{**}$$

(Here, N_v denotes the valuation of the discriminant of E_v and log denotes the p-adic logarithm.) This is correct. However, it is not the case that all of the values (**) necessarily occur.

With a careful analysis of which values do occur, it is possible to define certain subsets of our sets W in terms of a classification of possible reduction behaviors of our elliptic curve and to obtain an equality in (*) after all; see Theorem 1.6 of Bianchi [2]. (See also Remark 2.6 of loc. cit. for more details on specific cases in which the correction above is needed, and others in which the original statement does hold.)

An extreme case of the failure to attain all of the values (**) occurs when $\mathcal{X}(\mathcal{O}_v)$ is empty. This possibility (for elliptic curves as above, and for the hyperbolic curves considered in the article more generally) leads to several inaccuracies throughout the paper. As Betts has pointed out, this even includes the *base* case $\mathcal{X} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ where $\mathcal{X}(\mathbb{Z}_2) = \emptyset$.

When $\mathcal{X}(O_v) = \emptyset$ at some finite place v, the Selmer schemes $\mathrm{Sel}^n(\mathcal{X})$ "with stringent local conditions", as defined in paragraph 2.7 of the article, are empty. Thus, for instance,

$$\mathrm{Sel}^n(\mathbb{P}^1 \setminus \{0, 1, \infty\}) = \emptyset$$

and Conjecture 3.1 holds for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ over $\mathrm{Spec}\,\mathbb{Z}$ for trivial reasons. Consequently, the discussion in §6 is only of interest if one drops the stringent local conditions and returns to the Selmer schemes as defined originally in [3, 4].

The possibility that Selmer schemes can be empty necessitates several other minor corrections. These were pointed out to us by Betts.

- Remark 2.6. The correct statement is that $\mathrm{Im}\, j_v \subset \{0\}$.
- Proof of the Proposition in §2.8. Where we write "... the condition of locally belonging to the image of j_v is actually the same as triviality at v", we actually only have the left-to-right implication, which is the only direction we need.
- The Lemma in §2.9. This is correct as stated; however we don't actually use the statement regarding unramified torsors in the subsequent Proposition and Corollary.
- §6.1. The cohomology group "$H^1_{\mathbb{Z}}(G, U_1)$" in the lower-left corner of the first square is not the same as the Selmer scheme $\mathrm{Sel}^1(\mathcal{X})$, which, as we've already said, is empty.

References

[1] Jennifer S. Balakrishnan, Ishai Dan-Cohen, Minhyong Kim, and Stefan Wewers. A non-abelian conjecture of Tate-Shafarevich type for hyperbolic curves. *Math. Ann.*, 372(1-2):369–428, 2018.

[2] Francesca Bianchi. Quadratic Chabauty for (bi)elliptic curves and Kim's conjecture. *Algebra Number Theory*, 14(9):2369–2416, 2020.

[3] Minhyong Kim. The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel. *Invent. Math.*, 161(3):629–656, 2005.

[4] Minhyong Kim. The unipotent Albanese map and Selmer varieties for curves. *Publ. Res. Inst. Math. Sci.*, 45(1):89–133, 2009.