MA 841 (Greatest hits in number theory)
Fall 2024
Instructor: Jennifer Balakrishnan
Time: TR 9:30 – 10:45 AM, CDS 365
Office hours: TR 11 AM – 12 PM, CDS 409

Course overview

This course will survey a selection of celebrated papers in number theory. Many of these articles are well known because they posed questions that resulted in foundational lines of investigation, produced a spectacular technical breakthrough, or gave new perspective on a well-studied problem. But reading a paper for the first time is not always linear. How do we read papers in mathematics? With the goal of developing this skill, we will prepare expository talks and articles based on some of the most exciting developments in our field.

As a group, we will first select a subset of papers from the list on the following pages. (If there are further suggestions, please let me know!)

For each paper, we will assign the following roles[1]:

Mathematician,
Experimentalist, and
Historian/Science Journalist,

with the "Mathematician" role split between a pair of students and the "Experimentalist" role split between two students, depending on the style and length of the paper considered.

For each paper, I will start by giving an overview with background and context. The Mathematician will present the key ideas in the paper, and the Experimentalist will report on related computations. The role of the Historian/Science Journalist will be to give (aspirationally) a *Quanta Magazine*-level write-up of the paper based on the presentations, possibly supplemented by further reading. Everyone is expected to read the assigned materials before each class and actively participate.

NB: Optimistically, we will cover one paper approximately every one to two weeks in this style, but, of course, this depends on the papers we choose. Many of these papers deserve to be the topic of entire standalone semester-long seminars on their own! Our goal is more modest: to understand how to take apart a paper quickly, make black boxes, figure out how the main ideas hold together, and as necessary, describe the tools needed to open the black boxes.

As for how the course will go, I will meet with the Mathematician(s) and Experimentalist(s)

[1]With all credit for this framework due to Nick Trefethen, who famously led a numerical analysis course at Cornell in this manner: https://people.maths.ox.ac.uk/trefethen/classic_papers.txt

before their presentation to help prepare and suggest experiments. During the classes each week, the lectures will be split among Mathematician(s), Experimentalist(s), and myself, with a partition appropriate for the level of the paper.

By the end of the last lecture on the paper, the Historian/Science Journalist will draft a short article based on the lectures, share it with all of us, and we will collectively offer feedback on the article. To facilitate this, I will maintain a Dropbox folder of all articles and other associated readings and shared CoCalc projects for the write-ups and experiments.

Evaluation

Each student will actively participate in the course. The course grade will be based on the presentations (40%), written expository articles (40%), and participation (20%).

Potential papers

1. M. Bhargava and A. Shankar, “Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves,” *Ann. of Math. (2)* 181 (2015), no. 1, 191–242.
2. B. Birch and P. Swinnerton-Dyer, “Notes on Elliptic Curves (II),” *J. Reine Angew. Math*. 218 (1965), 79–108.
3. R. F. Coleman. "Effective Chabauty." *Duke Math. J*. 52 (3) 765 - 770, September 1985.
4. P. Deligne, “La conjecture de Weil. I.,” *Inst. Hautes Études Sci. Publ. Math.* No. 43 (1974), 273–307.
5. N. Elkies, “The existence of infinitely many supersingular primes for every elliptic curve over **Q**,” *Invent. Math.* 89 (1987) no. 3, 561–567.
6. G. Faltings[2], “Endlichkeitssätze für abelsche Varietäten über Zahlkörpern,” [Finiteness theorems for abelian varieties over number fields] *Invent. Math*. 73 (1983), no. 3, 349–366.
7. J.-M. Fontaine, “Il n'y a pas de variété abélienne sur **Z**,” [There are no abelian varieties over **Z**.] *Invent. Math*. 81 (1985), no. 3, 515–38.
8. T. Honda, “Isogeny classes of abelian varieties over finite fields,” *J. Math. Soc. Japan*, vol number 1-2 (1968): 83-95, with J. Tate, “Endomorphisms of abelian varieties over finite fields,” *Invent. Math.,* **2** (1966): 134-144. and J. Tate, “Classes d’isogénie des variétés abéliennes sur un corps fini,” *Séminaire N. Bourbaki*, 1971, exp. no 352, 95-110.
9. S. Kamienny and B. Mazur (with an appendix by A. Granville), “Rational torsion of prime order in elliptic curves over number fields, ” *Astérisque*. 228: 81–100 (1995).
10. H. Lenstra, “Factoring integers with elliptic curves,” *Ann. of Math. (2)* Vol. 126, No. 3 (Nov., 1987), 649–673.
11. B. Mazur (with an appendix by D. Goldfeld), “Rational isogenies of prime degree,” *Invent. Math.* 44 (1978), no. 2, 129–162.

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[2] We will likely instead use parts of *Rational points.* Third edition. Papers from the seminar held at the Max-Planck-Institut für Mathematik, Bonn, 1983/1984. Edited by Faltings and Wüstholz. Aspects of Mathematics, E6. *Friedr. Vieweg & Sohn, Braunschweig; distributed by Heyden & Son, Inc., Philadelphia, PA,* 1992. x+311.

12. B. Mazur and J. Tate, "Points of order 13 on elliptic curves," *Invent. Math.* 22 (1973/74), 41–49.
13. B. Mazur, J. Tate, and J. Teitelbaum. On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent Math* **84,** 1–48 (1986).
14. L. Merel, "Bornes pour la torsion des courbes elliptiques sur les corps de nombres," [Bounds for the torsion of elliptic curves over number fields]. *Invent. Math*. 124: (1996), no. 1, 437–449.
15. B. Poonen and M. Stoll, The Cassels-Tate pairing on polarized abelian varieties. *Ann. of Math.* (2) 150 (1999), no. 3, 1109–1149.
16. K. Ribet, "A modular construction of unramified p-extensions of $\mathbf{Q}(\mu_p)$," *Invent. Math.* 34 (1976), no. 3, 151–162.
17. J. Tate, "Fourier analysis in number fields and Hecke's zeta functions," Ph.D. thesis, Princeton, 1950.
18. A. Weil, "Numbers of solutions of equations in finite fields," *Bull. Amer. Math*. Soc. **55** (1949), 497-508.

Suggested in class (September 3)

19. B. Mazur, "Modular curves and the Eisenstein ideal" *Publications mathématiques de l'I.H.É.S.,* tome 47 (1977), p. 33-186.
20. P. Deligne, "Formes modulaires et représentations *l*-adiques." [Modular forms and l-adic representations] *Séminaire Bourbaki*. Vol. 1968/69: Exposés 347–363, Exp. No. 355, 139–172, Lecture Notes in Math., **175**, Springer-Verlag, Berlin, 1971.
21. A Scholl, "Motives for modular forms," *Invent. Math*. **100**(1990), no.2, 419–430.
22. H. M. Stark, "Values of L-functions at s = 1. I. L-functions for quadratic forms", *Advances in Math.* **7** (1971), 301—343. (+ parts II, III, IV)
23. B. H. Gross and D. Zagier, "Heegner points and derivatives of L-series," *Invent. Math.* **84** (1986), no. 2, 225—320. OR some chapters from H. Darmon's *Rational points on modular elliptic curves,* CBMS Reg. Con. Ser. Math., 101, AMS, Providence, RI, 2004. xii+129 pp.

We will vote among the following on Thursday, September 5:

- R. F. Coleman. "Effective Chabauty." *Duke Math. J*. 52 (3) 765 - 770, September 1985.
- G. Faltings[3], "Endlichkeitssätze für abelsche Varietäten über Zahlkörpern," [Finiteness theorems for abelian varieties over number fields] *Invent. Math*. 73 (1983), no. 3, 349–366.
- J.-M. Fontaine, "Il n'y a pas de variété abélienne sur **Z**," [There are no abelian varieties over **Z.**] *Invent. Math*. 81 (1985), no. 3, 515–38.
- T. Honda, "Isogeny classes of abelian varieties over finite fields," *J. Math. Soc. Japan*, vol number 1-2 (1968): 83-95, with J. Tate, "Endomorphisms of abelian varieties over finite

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fields," *Invent. Math.,* **2** (1966): 134-144. and J. Tate, "Classes d'isogénie des variétés abéliennes sur un corps fini," *Séminaire N. Bourbaki*, 1971, exp. no 352, 95-110.

- B. Mazur, J. Tate, and J. Teitelbaum. On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent Math* **84,** 1–48 (1986).
- K. Ribet, "A modular construction of unramified p-extensions of $\mathbf{Q}(\mu_p)$," *Invent. Math.* 34 (1976), no. 3, 151–162.
- P. Deligne, "Formes modulaires et représentations l-adiques." [Modular forms and l-adic representations] *Séminaire Bourbaki*. Vol. 1968/69: Exposés 347–363, Exp. No. 355, 139–172, Lecture Notes in Math., **175**, Springer-Verlag, Berlin, 1971.
- A Scholl, "Motives for modular forms," *Invent. Math*. **100**(1990), no.2, 419–430.
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Schedule and deadlines

September 3, 2024: Introduction to the course, organizational discussion
Note: If there is a paper you'd like to see that's not on the list, please feel free to nominate it during the discussion in class on September 3rd.

September 5, 2024: Choosing papers, assigning roles

September 10, 2024:
September 12, 2024:

September 17, 2024:
September 19, 2024:

September 24, 2024:
September 26, 2024:

October 1, 2024:
October 3, 2024:

October 8, 2024:
October 10, 2024:

October 17, 2024:
October 22, 2024:
October 24, 2024:

October 29, 2024:
October 31, 2024:

November 5, 2024:
November 7, 2024:

November 12, 2024:
November 14, 2024:

November 19, 2024:
November 21, 2024:

November 26, 2024: No class, JB away

December 3, 2024: Paper-writing wrap up and live editing
December 5, 2024: Paper-writing wrap up and live editing

December 10, 2024: Course wrap up