MA 842 (Greatest hits in arithmetic geometry)
Spring 2025
Instructor: Jennifer Balakrishnan
Time: TR 2:00 – 3:15 PM, CDS 424
Office hours: TR 3:15 – 4:15 PM, CDS 409

Course overview

This course will survey a selection of celebrated papers in arithmetic geometry. Many of these articles are well known because they posed questions that resulted in foundational lines of investigation, produced a spectacular technical breakthrough, or gave new perspective on a well-studied problem. But reading a paper for the first time is not always linear. How do we read papers in mathematics? With the goal of developing this skill, we will prepare expository talks and articles based on some of the most exciting developments in our field.

As a group, we will first select a subset of papers from the list on the following pages. (If there are further suggestions, please let me know!)

For each paper, we will assign the following roles[1]:

Mathematician,
Experimentalist, and
Writer,

with the "Mathematician" role split among several students, depending on the length of the paper.

For each paper, I will start by giving an overview with background and context. The Mathematician(s) will present the key ideas in the paper, and the Experimentalist will report on related computations. The role of the Writer will be to give (aspirationally) a *Quanta Magazine*-level write-up of the paper based on the presentations, possibly supplemented by further reading. Everyone is expected to read the assigned materials before each class and actively participate.

NB: Optimistically, we will cover one paper approximately every 2 to 4 weeks in this style, but, of course, this depends on the papers we choose. Many of these papers deserve to be the topic of entire standalone semester-long seminars on their own! Our goal is more modest: to understand how to take apart a paper quickly, make black boxes, figure out how the main ideas hold together, and as necessary, describe the tools needed to open the black boxes.

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[1]With all credit for this framework due to Nick Trefethen, who famously led a numerical analysis course at Cornell in this manner: https://people.maths.ox.ac.uk/trefethen/classic_papers.txt

As for how the course will go, I will meet with the Mathematician(s) and Experimentalist before their presentation to help prepare and suggest experiments. During the classes each week, the lectures will be split among Mathematician(s), Experimentalist, and myself, with a partition appropriate for the level of the paper.

By the end of the last lecture on the paper, the Writer will draft a short article based on the lectures, share it with all of us, and we will collectively offer feedback on the article. To facilitate this, I will maintain a Dropbox folder of all articles and other associated readings and shared CoCalc projects for the write-ups and experiments.

Evaluation

Each student will actively participate in the course. The course grade will be based on the presentations (40%), written expository articles (40%), and participation (20%).

Potential papers

1. M. Bhargava and A. Shankar, "Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves," *Ann. of Math. (2)* 181 (2015), no. 1, 191–242.
2. B. Birch and P. Swinnerton-Dyer, "Notes on Elliptic Curves (II)," *J. Reine Angew. Math*. 218 (1965), 79–108.
3. P. Deligne, "La conjecture de Weil. I.," *Inst. Hautes Études Sci. Publ. Math.* No. 43 (1974), 273–307.
4. N. Elkies, "The existence of infinitely many supersingular primes for every elliptic curve over **Q**," *Invent. Math.* 89 (1987) no. 3, 561–567.
5. J.-M. Fontaine, "Il n'y a pas de variété abélienne sur **Z**," [There are no abelian varieties over **Z**.] *Invent. Math*. 81 (1985), no. 3, 515–38.
6. B. H. Gross and D. Zagier, "Heegner points and derivatives of *L*-series," *Invent. Math.* **84** (1986), no. 2, 225—320.
7. T. Honda, "Isogeny classes of abelian varieties over finite fields," *J. Math. Soc. Japan*, vol number 1-2 (1968): 83-95, with J. Tate, "Endomorphisms of abelian varieties over finite fields," *Invent. Math.,* **2** (1966): 134-144. and J. Tate, "Classes d'isogénie des variétés abéliennes sur un corps fini," *Séminaire N. Bourbaki*, 1971, exp. no 352, 95-110.
8. S. Kamienny and B. Mazur (with an appendix by A. Granville), "Rational torsion of prime order in elliptic curves over number fields, " *Astérisque*. 228: 81–100 (1995).
9. B. Mazur, "Modular curves and the Eisenstein ideal" *Publications mathématiques de l'I.H.É.S.,* tome 47 (1977), p. 33-186.
10. B. Mazur (with an appendix by D. Goldfeld), "Rational isogenies of prime degree," *Invent. Math.* 44 (1978), no. 2, 129–162.
11. B. Mazur and J. Tate, "Points of order 13 on elliptic curves," *Invent. Math.* 22 (1973/74), 41–49.
12. L. Merel, "Bornes pour la torsion des courbes elliptiques sur les corps de nombres," [Bounds for the torsion of elliptic curves over number fields]. *Invent. Math*. 124: (1996), no. 1, 437–449.

13. B. Poonen and M. Stoll, “The Cassels-Tate pairing on polarized abelian varieties,” *Ann. of Math.* (2) 150 (1999), no. 3, 1109–1149.
14. K. Ribet, “A modular construction of unramified p-extensions of $\mathbf{Q}(\mu_p)$,” *Invent. Math.* 34 (1976), no. 3, 151–162.
15. A. Weil, “Numbers of solutions of equations in finite fields,” *Bull. Amer. Math*. Soc. **55** (1949), 497-508.

Suggested in class (January 21)

16. S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives. The Grothendieck Festschrift, Vol. I, 333–400. Progr. Math., 86, Birkhäuser Boston, Inc., Boston, MA, 1990
17. J.-P. Serre, “Répartition asymptotique des valeurs propres de l'opérateur de Hecke Tp.” (French)[Asymptotic distribution of the eigenvalues of the Hecke operator Tp] J. Amer. Math. Soc. **10** (1997), no. 1, 75–102.
18. P. Deligne and G. Lusztig, “Representations of reductive groups over finite fields.” Ann. of Math. (2) **103** (1976), no. 1, 103–161.
19. S. Bloch, “Algebraic cycles and values of L-functions.” J. Reine Angew. Math. **350** (1984), 94–108.
20. W. Casselman, “The unramified principal series of p-adic groups. I. The spherical function.” Compositio Math. **40** (1980), no. 3, 387–406.
21. W. Casselman and J. Shalika, “The unramified principal series of p-adic groups. II. The Whittaker function.” Compositio Math. **41** (1980), no. 2, 207–231.
22. B. Green and T. Tao, “The primes contain arbitrarily long arithmetic progressions.” Ann. of Math. (2) **167** (2008), no. 2, 481–547.
23. B. Mazur and A. Wiles, “Analogies between function fields and number fields.” Amer. J. Math. **105** (1983), no. 2, 507–521.
24. F. Herzig, “The classification of irreducible admissible mod p representations of a p-adic GLn.” Invent. Math. **186** (2011), no. 2, 373–434.
25. V. G. Drinfel'd, “Elliptic modules.” (Russian) Mat. Sb. (N.S.) **94(136)** (1974), 594–627, 656.
26. H. Pasten, “The largest prime factor of n^2+1 and improvements on subexponential ABC.” Invent. Math. **236** (2024), no. 1, 373–385.
27. B. H. Gross and K. Keating, “On the intersection of modular correspondences.” Invent. Math. **112** (1993), no. 2, 225–245.

We will vote among the following on Thursday, January 23:

1. B. H. Gross and D. Zagier, “Heegner points and derivatives of *L*-series,” *Invent. Math.* **84** (1986), no. 2, 225—320. https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=833192
2. B. Mazur, “Modular curves and the Eisenstein ideal” *Publications mathématiques de l'I.H.É.S.,* tome 47 (1977), p. 33-186. https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=488287

3. B. Mazur and J. Tate, "Points of order 13 on elliptic curves," *Invent. Math.* 22 (1973/74), 41–49.
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=347826
4. J.-P. Serre, "Répartition asymptotique des valeurs propres de l'opérateur de Hecke Tp." (French)[Asymptotic distribution of the eigenvalues of the Hecke operator Tp] J. Amer. Math. Soc. **10** (1997), no. 1, 75–102.
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=1396897
5. P. Deligne and G. Lusztig, "Representations of reductive groups over finite fields." Ann. of Math. (2) **103** (1976), no. 1, 103–161.
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=393266
6. S. Bloch, "Algebraic cycles and values of L-functions." J. Reine Angew. Math. **350** (1984), 94–108.
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=743535
7. B. Mazur and A. Wiles, "Analogies between function fields and number fields." Amer. J. Math. **105** (1983), no. 2, 507–521.
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=701567
8. V. G. Drinfel'd, "Elliptic modules." (Russian) Mat. Sb. (N.S.) **94(136)** (1974), 594–627, 656.
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=384707
9. H. Pasten, "The largest prime factor of n^2+1 and improvements on subexponential ABC." Invent. Math. **236** (2024), no. 1, 373–385.
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=4712867
10. T. Honda, "Isogeny classes of abelian varieties over finite fields," *J. Math. Soc. Japan*, vol number 1-2 (1968): 83-95, with J. Tate, "Endomorphisms of abelian varieties over finite fields," *Invent. Math.,* **2** (1966): 134-144. and J. Tate, "Classes d'isogénie des variétés abéliennes sur un corps fini," *Séminaire N. Bourbaki*, 1971, exp. no 352, 95-110.
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=229642
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=206004
11. S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives. The Grothendieck Festschrift, Vol. I, 333–400. Progr. Math., 86, Birkhäuser Boston, Inc., Boston, MA, 1990
https://mathscinet-ams-org.ezproxy.bu.edu/mathscinet/article?mr=1086888

Schedule and deadlines

January 21, 2025: Introduction to the course, organizational discussion
Note: If there is a paper you'd like to see that's not on the list, please feel free to nominate it during the discussion in class today.
January 23, 2025: Choosing papers, assigning roles

Mazur [W: Kate]

January 28, 2025: JB
January 30, 2025: Ashutosh

February 4, 2025: Zecheng
February 6, 2025: Xinyu

February 11, 2025: Liqiang
February 13, 2025: Ashutosh

February 18, 2025: BU Monday, no class
February 20, 2025: JB out of town, no class

February 25, 2025: Zecheng: Experiments
February 27, 2025: Writing discussion with Kate's draft ready

Bloch [W: Xinyu]

March 4, 2025: JB
March 6, 2025: Matt

March 11, 2025: Spring break, no class
March 13, 2025: Spring break, no class

March 18, 2025: Tate conference (Harvard); class rescheduled
March 20, 2025: Tate conference (Harvard); class rescheduled

March 25, 2025: Kate: Experiments
March 27, 2025: Writing discussion with Xinyu's draft ready

April 1, 2025: JB out of town; class rescheduled
April 3, 2025: JB out of town; class rescheduled

Gross—Zagier [W: JB]

April 8, 2025: JB
April 10, 2025: Liqiang

April 15, 2025: Matt
April 17, 2025: Xinyu

April 22, 2025: Ashutosh
April 24, 2025: Matt

April 29, 2025: Liqiang: Experiments
May 1, 2025: Writing discussion with JB's draft ready

Serre [W: Zecheng]

May 5, 2025: JB
May 6, 2025: Kate
May 7, 2025: Jacksyn: Experiments
May 8, 2025: Writing discussion with Zecheng's draft ready