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What is on today

1 Limits at infinity

Briggs-Cochran-Gillett §§2.5 – 2.6 pp. 88–112.

1.1 Limits at infinity and horizontal asymptotes

Example 1 (§2.5 **Ex. 10)** *Evaluate*

$$\lim_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)$$

Definition 2 (Limits at infinity and horizontal asymptotes) *If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write*

$$\lim_{x\to\infty} f(x) = L,$$

and we say that the ***limit of $f(x)$ as x approaches infinity is L****.*

In this case the line $y = L$ is a ***horizontal asymptote of f****.*

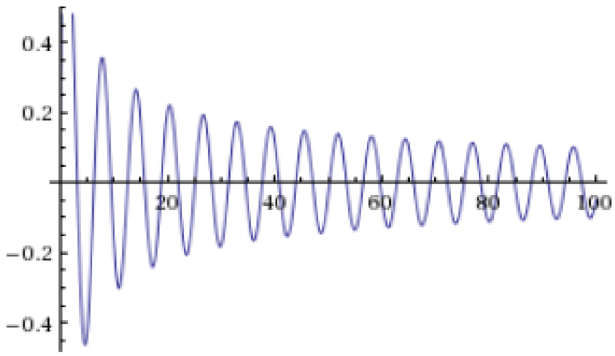
The limit at negative infinity $\lim_{x\to-\infty} f(x) = M$ is defined analogously. When it exists $y = M$ is also called a horizontal asymptote.

Example 3 (§2.5 **Ex. 28)** *Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Then give the horizontal asymptotes of f (if any).*

$$f(x) = \frac{4x^2-7}{8x^2+5x+2}$$

Remark 4 *Note that the graph of f* ***can*** *intersect its horizontal asymptote!*

Example 5 *Consider the function* $f(x) = \frac{\sin x}{\sqrt{x}}$*:*



What is the horizontal asymptote here?

1.2 Infinite limits at infinity

Definition 6 (Infinite limits at infinity) *If f becomes arbitrarily large as x becomes arbitrarily large, then we write* $\lim_{x\to\infty} f(x) = +\infty$. *The limits* $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = +\infty$ *and* $\lim_{x\to-\infty} f(x) = -\infty$ *are defined similarly.*

Example 7 (§2.5 **Ex. 24)** *Determine the limit* $\lim_{x\to-\infty}(2x^{-8} + 4x^3)$.

1.3 End behavior

Theorem 8 (End behavior of functions) *Let n, m be a positive integers and* $p(x) = a_m x^m + \ldots + a_1 x + a_0$, $q(x) = b_n x^n + \ldots + b_1 x + b_0$ *polynomials with* $a_m, b_n \neq 0$.

1. $\lim_{x\to\pm\infty} x^n = \infty$ *when n is even;*
2. $\lim_{x\to\infty} x^n = \infty$ *and* $\lim_{x\to-\infty} x^n = -\infty$ *when n is odd;*
3. $\lim_{x\to\infty} x^{-n} = \lim_{x\to\infty} \frac{1}{x^n} = 0$;
4. $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_m x^m$;
5. *If $m < n$ (degree of numerator less than that of denominator) then* $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = 0$;
6. *If $m = n$ (degree of numerator equal to that of denominator) then* $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \frac{a_m}{b_n}$;
7. *If $m > n$ (degree of numerator greater than that of denominator) then* $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \infty$ *or* $-\infty$;

8. $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$, $\lim_{x\to\infty} e^{-x} = 0$, $\lim_{x\to-\infty} e^{-x} = \infty$;
9. $\lim_{x\to 0^+} \ln(x) = -\infty$, $\lim_{x\to\infty} \ln(x) = \infty$;

Example 9 (§2.5 Ex. 42) *Determine the given limits and the give the horizontal asymptotes of f (if any).*

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1}$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x+1}$$

2 Continuity

Informally, we say that a function f is *continuous* at a if the graph of f does not have a hole or break at a (that is, if the graph near a can be drawn without lifting the pencil). If a function is not continuous at a, then a is a point of discontinuity. We have already seen a number of functions that are continuous:

Theorem 10 *Polynomial functions are continuous for all x. A rational function (a function of the form p/q where p, q are polynomials) is continuous for all x for which $q(x) \neq 0$.*

The informal description of continuity above is sufficient for determining the continuity of simple functions, but it is not precise enough to deal with more complicated functions.

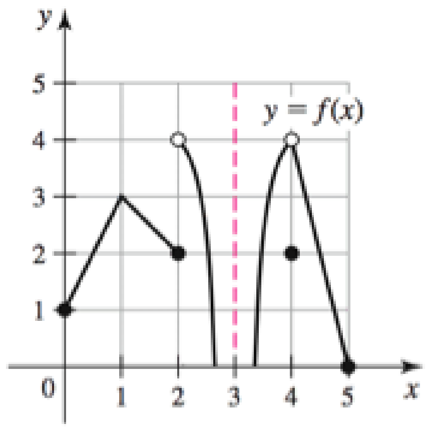
Definition 11 (Continuity at a(n interior) point) *A function f is continuous at a (where a is not given as an endpoint of an interval) if $\lim_{x\to a} f(x) = f(a)$. If f is not continuous at a, then a is a point of discontinuity.*

2.1 Continuity checklist

In order for f to be continuous at a as in the definition above, the following three conditions must hold:

1. $f(a)$ is defined (a is in the domain of f).
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Example 12 (§2.6 **Ex. 12**) *Determine the points at which f has discontinuities.*



We have some helpful results about composite functions:

Theorem 13 (Continuity of composite functions at a point) *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.*

Theorem 14 (Limits of composite functions)

1. *If g is continuous at a and f is continuous at $g(a)$, then $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right)$.*
2. *If $\lim\limits_{x\to a} g(x) = L$ and f is continuous at L, then $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right)$.*

2.2 Continuity on an interval

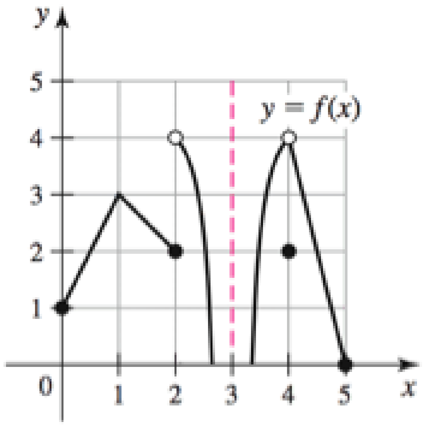
When we talk about continuity of functions, to be more precise, we ought to specify the interval of the function we are considering. A function is *continuous on an interval* if it is continuous at every point in that interval. We must be careful about endpoints of intervals:

Definition 15 (Continuity at endpoints) *A function f is continuous from the left (or left-continuous) at a is $\lim\limits_{x\to a^-} f(x) = f(a)$, and f is continuous from the right (or right-continuous) at a if $\lim\limits_{x\to a^+} f(x) = f(a)$.*

Combining the definitions of left-continuous and right continuous with the definition of continuity at a point (Definition 11), we define what it means for a function to be continuous on an interval:

Definition 16 (Continuity on an interval) *A function f is continuous on an interval I if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the endpoints.*

Example 17 (§2.6 **Ex. 12**) *Give the intervals where f is continuous.*



Example 18 (§2.6 **Ex. 18**) *Determine whether f is continuous at $a = 3$.*

$$f(x) = \begin{cases} \frac{x^2-4x+3}{x-3} & \text{if} \quad x \neq 3 \\ 2 & \text{if} \quad x = 3 \end{cases}$$

Example 19 (§2.6 **Ex. 39**) *Let*

$$f(x) = \begin{cases} 2x & \text{if} \quad x < 1 \\ x^2 + 3x & \text{if} \quad x \geq 1 \end{cases}$$

1. *Is f continuous at 1?*
2. *Is f continuous from the left or right at 1?*
3. *State the interval(s) of continuity.*

2.3 Intermediate Value Theorem

Continuity plays an important role in the following theorem:

Theorem 20 (Intermediate Value Theorem) *Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.*