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What is on today

Briggs-Cochran-Gillett §§3.1 – 3.2 pp. 126–143

1 Derivatives

In this section, we revisit the idea of finding the slope of a line tangent to a curve. Today we will

- Identify the slope of the tangent line with the *instantaneous rate of change* of a function
- Study the slopes of the tangent lines as they change along a curve (these slopes are the values of a new function called the *derivative*)

1.1 Rate of change and the slope of the tangent line

Definition 1 (Rate of change and the slope of the tangent line) *The average rate of change in f on the interval $[a, x]$ is the slope of the corresponding secant line:*

$$m_{sec} = \frac{f(x) - f(a)}{x - a}.$$

The instantaneous rate of change in f at a is

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

which is also the slope of the tangent line at $(a, f(a))$, provided this limit exists. The tangent line is the unique line through $(a, f(a))$ with slope m_{tan}. Its equation is $y - f(a) = m_{tan} \cdot (x - a)$.

Here is an alternative definition:

Definition 2 (Rate of change and the slope of the tangent line) *The average rate of change in f on the interval $[a, a+h]$ is the slope of the corresponding secant line:*

$$m_{sec} = \frac{f(a+h) - f(a)}{h}.$$

The instantaneous rate of change in f at a is

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

which is also the slope of the tangent line at $(a, f(a))$, provided this limit exists.

Example 3 (§3.1 Ex. 10) *Let $f(x) = -3x^2 - 5x + 1$ and consider $P = (1, -7)$. Use Definition 1 to find the slope of the line tangent to the graph of f at P. Determine an equation of the tangent line at P.*

1.2 The derivative function

Definition 4 *The derivative of f is the function*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

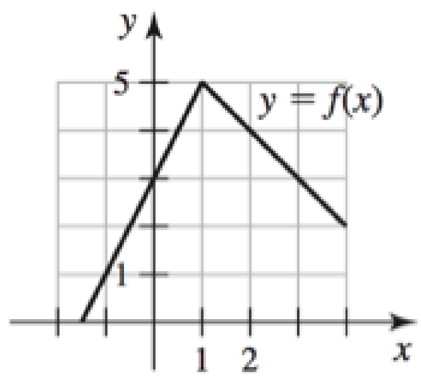
provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say that f is differentiable at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

Example 5 (§3.1 Ex. 30, 34) *For the following functions and values of a, find $f'(a)$. Determine an equation of the line tangent to the graph of f at the point $(a, f(a))$ for the given value of a.*

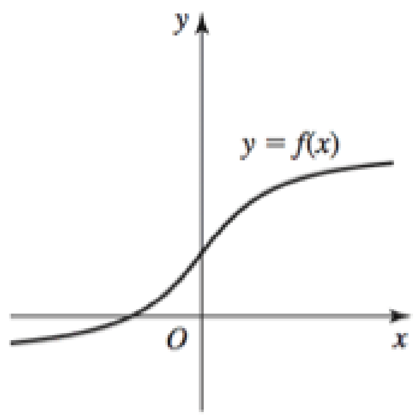
1. $f(x) = 2x^3, a = 10$
2. $f(x) = \sqrt{3x}, a = 12$

Having defined the derivative, we now explore how the graphs of a function and its derivative are related.

Example 6 (§3.2 **Ex. 6**) *Use the graph of f to sketch a graph of f':*



Example 7 (§3.2 **Ex. 12**) *Reproduce the graph of f and then sketch a graph of f' on the same axes.*



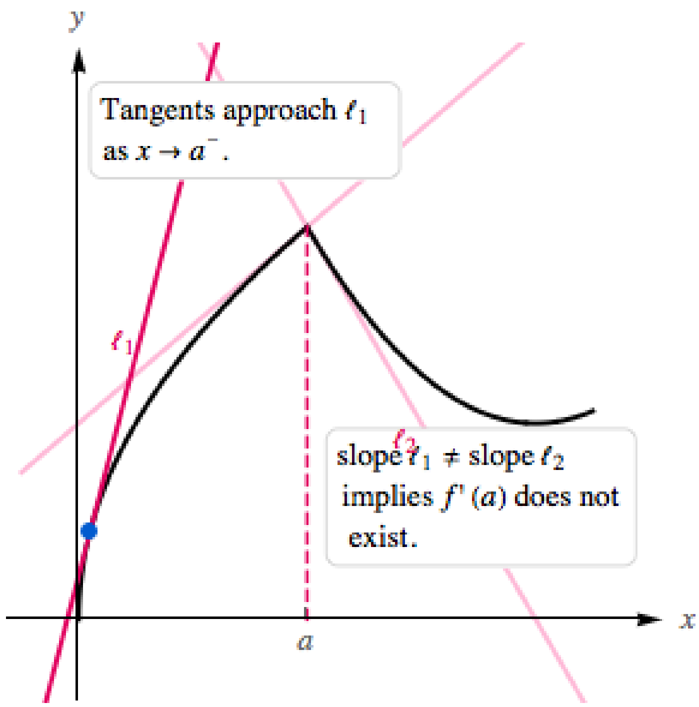
1.3 Differentiability and continuity

Theorem 8 (Differentiable implies continuous) *If f is differentiable at a, then f is continuous at a.*

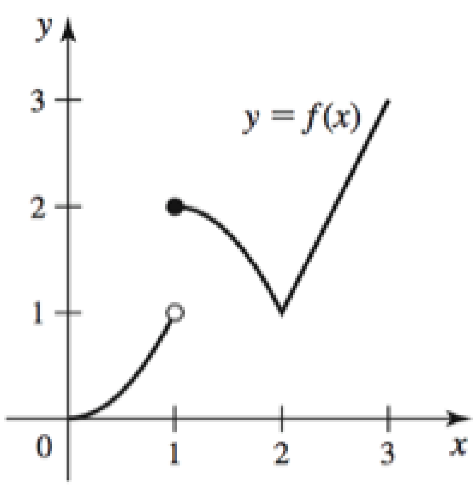
This can be stated in another way:

Theorem 9 (Not continuous implies not differentiable) *If f is not continuous at a, then f is not differentiable at a.*

Be careful: it might be tempting to read more into Theorem 8 than what it actually states. Note that if f is continuous at a point, f is *not necessarily* differentiable at that point. Here is one such example:



Example 10 (§3.2 **Ex. 15**) *Use the graph of f in the figure to do the following.*



1. *Find the values of x in (0,3) at which f is not continuous*
2. *Find the values of x in (0,3) at which f is not differentiable.*
3. *Sketch a graph of f'.*

2 Rules of Differentiation

Briggs-Cochran-Gillett §3.3 pp. 144 – 147

2.1 The constant, power, constant multiple and sum rules

Theorem 11 (First Differentiation Rules) *Let c be a constant, n a positive integer and f and g differentiable functions.*

- *Constant rule:* $\dfrac{d}{dx}(c) = 0$.
- *Power rule:* $\dfrac{d}{dx}(x^n) = nx^{n-1}$.
- *Constant multiple rule:* $\dfrac{d}{dx}(cf(x)) = cf'(x)$.
- *Sum rule:* $\dfrac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$.

Example 12 (§3.3 Ex. 20) *Find the derivative of* $g(x) = 6x^5 - x$.

Example 13 (§3.3 Ex. 22) *Find the derivative of* $f(t) = 6\sqrt{t} - 4t^3 + 9$.

Example 14 (§3.3 Ex. 26) *Find the derivative of* $g(r) = (5r^3 + 3r + 1)(r^2 + 3)$ *by first expanding the expression. Simplify your answer.*