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What is on today

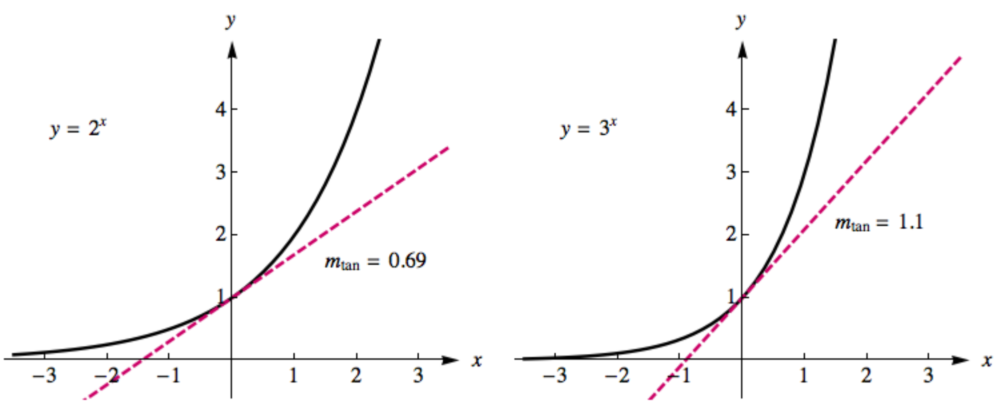
1 Rules of differentiation, continued

Briggs-Cochran-Gillett §3.3 pp. 148-152

1.1 Derivative of e^x

Definition of e

Exponential functions b^x look like



The number e can be defined as the base needed in the exponential function to get the slope of the tangent to the graph at $x = 0$ equal to 1.

✌ See interactive Figure 3.25 of book!

Definition 1. *e^x is the exponential function such that the slope of the tangent to the graph at $x = 0$ is 1, i.e.,*

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Derivative of $\mathbf{e^x}$

Theorem 2. *The function $f(x) = e^x$ is differentiable for all real numbers x, and*

$$\frac{d}{dx} e^x = e^x.$$

Example 3 (§3.3 Ex. 38)**.** *Find an equation of the tangent line to $y = \dfrac{e^x}{4} - x$ at $a = 0$. Then use a graphing utility to graph the curve and the tangent line on the same set of axes.*

1.2 Higher order derivatives

Definition 4. *Assuming the function $f(x)$ can be differentiated as often as necessary, the* ***second derivative*** *of f is*

$$f''(x) = \frac{d}{dx}(f'(x)).$$

For integers $n \geq 2$, the **n*th derivative*** *of f is*

$$f^n(x) = \frac{d}{dx}(f^{(n-1)}(x)).$$

Example 5 (§3.3 Ex. 46 and 48)**.** *Find $f'(x)$, $f''(x)$, and $f'''(x)$ for the following functions:*

1. $f(x) = 3x^2 + 5e^x$
2. $f(x) = 10e^x$

2 The product and quotient rules

Briggs-Cochran-Gillett §3.4 pp. 153-162

We saw in the last class that the derivative of a sum of functions is the sum of the derivatives. So you might wonder about the derivative of a product of functions: is it the product of the derivatives? Consider $f(x) = x^3$ and $g(x) = x^4$; in this case, $\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(x^7) = 7x^6$ but $f'(x)g'(x) = 3x^2 \cdot 4x^3 = 12x^5$. Thus $\frac{d}{dx}(f(x)g(x)) \neq f'(x)g'(x)$ and likewise, the derivative of a quotient is not the quotient of the derivatives.

Today we discuss the rules for differentiating products and quotients of functions and extend a few rules from last class.

2.1 Product rule

Theorem 6 (Product rule)**.** *If f and g are differentiable at x, then*

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

2.2 Quotient rule

Theorem 7 (Quotient rule)**.** *If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of $\frac{f}{g}$ at x exists and*

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

2.3 Extended power rule

Theorem 8. *If n is any integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$.*

2.4 Exponentials

Theorem 9. *For real numbers k, we have $\frac{d}{dx}(e^{kx}) = ke^{kx}$.*

2.5 Examples

Now we practice applying these rules.

Example 10 (§3.4 Ex. 8, 12, 24)**.** *Compute the derivative of the following functions:*

1. $g(x) = 6x - 2xe^x$
2. $f(x) = (1 + \frac{1}{x^2})(x^2 + 1)$
3. $f(x) = e^{-x}\sqrt{x}$

Example 11 (§3.4 Ex. 32). *Let* $y = \frac{x^2-2ax+a^2}{x-a}$, *where* a *is a constant.*

1. *Use the quotient rule to find the derivative of the given function. Simplify your result.*
2. *Find the derivative by first simplifying the function. Verify that your answer agrees with part (1).*

Example 12 (§3.4 Ex. 40, 48). *Find the derivative of the following functions:*

1. $y = \frac{w^4+5w^2+w}{w^2}$.
2. $f(x) = (1-2x)e^{-x}$

Example 13 (§3.4 Ex. 54). *A* \$200 *investment in a savings account grows according to* $A(t) = 200e^{0.0398t}$ *for* $t \geq 0$, *where* t *is measured in years.*

1. *Find the balance of the account after 10 years.*
2. *How fast is the account growing (in dollars/year) at* $t = 10$*?*
3. *Use your answers to parts (1) and (2) to write the equation of the line tangent to the curve* $A = 200e^{0.0398t}$ *at the point* $(10, A(10))$.

Example 14 (§3.4 Ex. 60). *Compute the derivative of* $h(x) = \frac{x+1}{x^2e^x}$.

Example 15 (§3.4 Ex. 72). *Suppose* $f(2) = 2$ *and* $f'(2) = 3$. *Let* $g(x) = x^2f(x)$ *and* $h(x) = \frac{f(x)}{x-3}$.

1. *Find an equation of the line tangent to* $y = g(x)$ *at* $x = 2$.
2. *Find an equation of the line tangent to* $y = h(x)$ *at* $x = 2$.