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What is on today

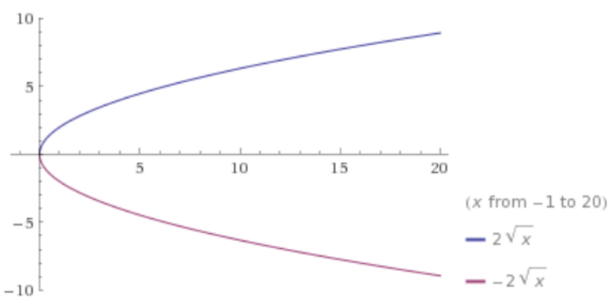
1 Implicit differentiation

Briggs-Cochran-Gillett §3.8 pp. 195 – 202

1.1 What it is and how to do it

Consider the curve defined by $y^2 = 4x$, for $x \in \mathbb{R}$. To represent it in the xy-plane we need to solve for x:

$$y^2 = 4x$$
$$\Leftrightarrow y = \pm\sqrt{4x}$$



So y is not a function of x, because for each value of x there are 2 values of y. But the original expression does define a curve in the plane which is the union of the graphs of the two functions $y = \sqrt{4x}$ and $y = -\sqrt{4x}$. We say in this case that y is **implicitly defined** by the expression $y^2 = 4x$.

If we want to find slopes of lines tangent to this curve we need to differentiate the original expression. In this example we could solve it for y and then differentiate. But in many examples this is not possible. So we want to **implicitly** differentiate the original expression to get an expression for dy/dx. We have

$$(y(x))^2 = 4x.$$

Hence

$$\frac{d}{dx}(y(x))^2 = \frac{d}{dx}(4x) \Leftrightarrow 2y(x)\frac{d}{dx}(y(x)) = 4 \Leftrightarrow \frac{d}{dx}(y(x)) = \frac{2}{y(x)}, \text{ if } y \neq 0.$$

Note that the derivative $\frac{d}{dx}(y(x))$ is not defined at $y = 0$: that corresponds to the point where the curve has a vertical tangent line, hence its slope is not defined. At any other point we can find the slope of the tangent line. For example, the slope of the tangent line at $(1, 2)$ (which is a point on the curve) will be

$$y'(2) = \frac{2}{y(1)} = \frac{2}{2} = 1.$$

1.2 Examples

Example 1 (§3.8 Ex. 12)**.** *Let* $\tan(xy) = x + y$*. Use implicit differentiation to find* dy/dx *and find the slope of the curve at the point* $(0, 0)$*.*

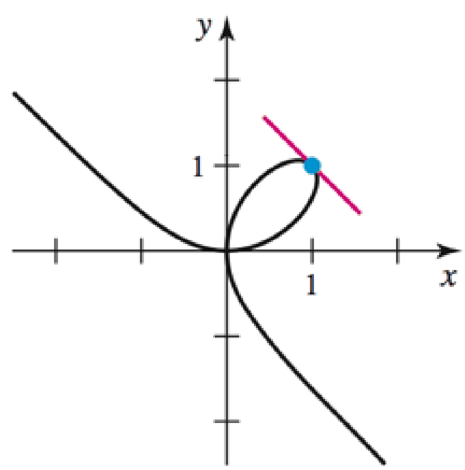
Example 2 (§3.8 Ex. 14, 20 and 24)**.** *Use implicit differentiation to find* dy/dx *for the following functions:*

a. $e^{xy} = 2y$

b. $(xy + 1)^3 = x - y^2 + 8$

c. $\sqrt{x + y^2} = \sin y$

Example 3 (§3.8 Ex. 28)**.** *Consider the curve defined by* $x^3 + y^3 = 2xy$*.*



Verify the the point $(1, 1)$ *lies on the curve and determine an equation of the line tangent to the curve at that point.*

Example 4 (§3.8 Ex. 44)**.** *Let* $y = \dfrac{x}{\sqrt[5]{x} + x}$. *Find* dy/dx.

Example 5 (§3.8 Ex. 48)**.** *Determine the slope of the curve* $(x+y)^{2/3} = y$ *at the point* $(4,4)$.

Example 6 (§3.8 Ex. 76)**.** *The lateral surface area of a cone of radius* r *and height* h *(the surface area excluding the base) is* $A = \pi r\sqrt{r^2+h^2}$.

a. *Find* dr/dh *for a cone with a lateral surface area of* $A = 1500\pi$.

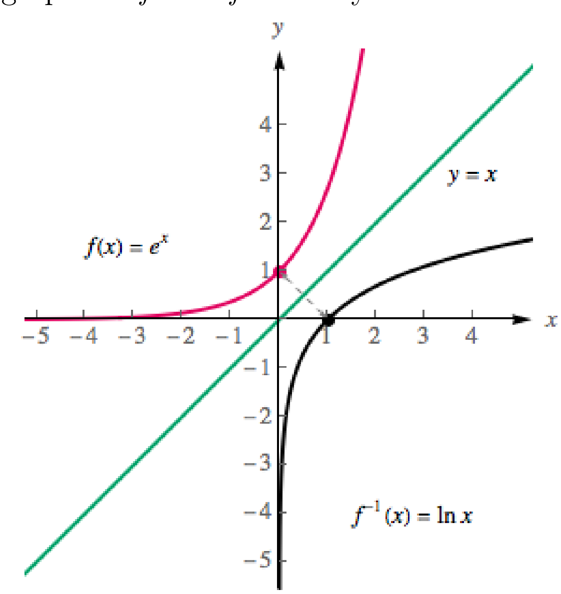
b. *Evaluate this derivative when* $r = 30$ *and* $h = 40$.

2 Derivatives of logarithmic and exponential functions

Briggs-Cochran-Gillett §3.9 pp. 203 – 213

2.1 Derivatives

Recall that the exponential function $f(x) = e^x$ is a one-to-one function on the interval $(-\infty, \infty)$. Thus it has an inverse, which is the natural logarithmic function $f^{-1}(x) = \ln(x)$. The graphs of f and f^{-1} are symmetric about the line $y = x$, as below.



We summarize the inverse properties for e^x and $\ln x$ below:

1. $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all x.
2. $y = \ln x$ if and only if $x = e^y$.
3. For real numbers x and $b > 0$, we have $b^x = e^{\ln b^x} = e^{x \ln b}$.

We can use implicit differentiation to calculate the derivative of $\ln x$. This yields the following:

Theorem 7 (Derivative of $\ln x$). *We have*

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \text{ for } x > 0 \text{ and } \frac{d}{dx}(\ln |x|) = \frac{1}{x} \text{ for } x \neq 0.$$

If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}(\ln |u(x)|) = \frac{u'(x)}{u(x)}.$$

The property $e^{x \ln b} = b^x$ results in the following theorem:

Theorem 8 (Derivative of b^x). *If $b > 0$ and $b \neq 1$, then for all x, we have*

$$\frac{d}{dx}(b^x) = b^x \ln b.$$

We can also use this to extend the power rule to all real exponents:

Theorem 9 (General power rule). *For real numbers p and for $x > 0$, we have*

$$\frac{d}{dx}(x^p) = px^{p-1}.$$

Furthermore, if u is a positive differentiable function on its domain, then

$$\frac{d}{dx}(u(x)^p) = p(u(x))^{p-1} \cdot u'(x).$$

We also use implicit differentiation to calculate the derivative of $\log_b x$:

Theorem 10 (Derivative of $\log_b x$). *If $b > 0$ and $b \neq 1$, then*

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}, \text{ for } x > 0 \text{ and } \frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b}, \text{ for } x \neq 0.$$

2.2 Exercises

Example 11 (§3.9 Ex. 12, 18, 20, 40). *Find the following derivatives.*

1. $\frac{d}{dx}(\ln 2x^8)$
2. $\frac{d}{dx}(\ln |x^2 - 1|)$
3. $\frac{d}{dx}(\ln(\cos^2 x))$
4. $\frac{d}{dx}(\ln(x^3 + 1)^\pi)$