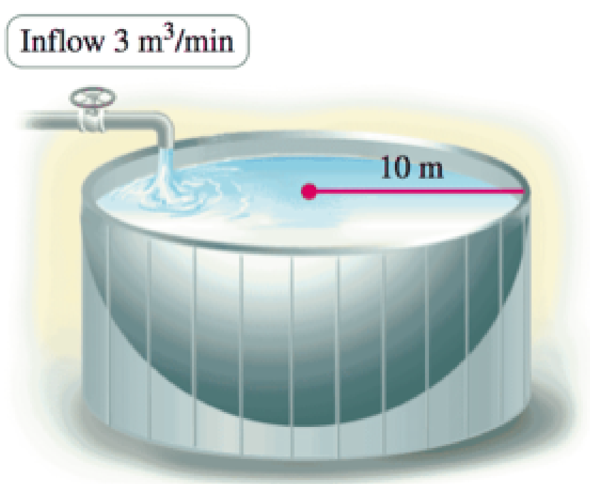
Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Related rates, wrap-up

Briggs-Cochran-Gillett §3.11 pp. 224-231

Example 1 (§3.11 Ex. 32). *A hemispherical tank with a radius of 10 m is filled from an inflow pipe at a rate of 3 m^3/min (see figure). How fast is the water level rising when the water level is 5 m from the bottom of the tank? (Hint: the volume of a cap of thickness h sliced from a sphere of radius r is $\pi h^2(3r - h)/3$.*



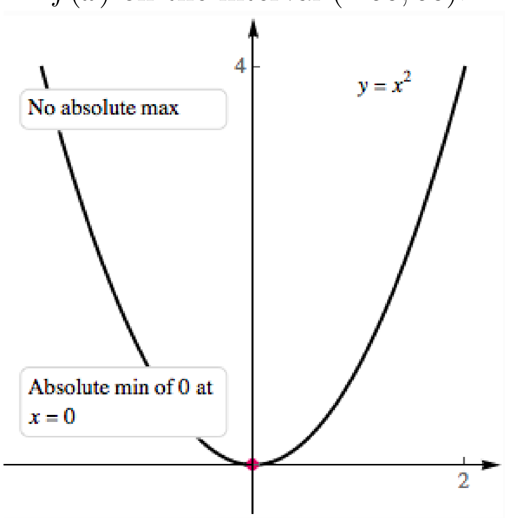
2 Maxima and minima

Briggs-Cochran-Gillett §4.1 pp. 236-245

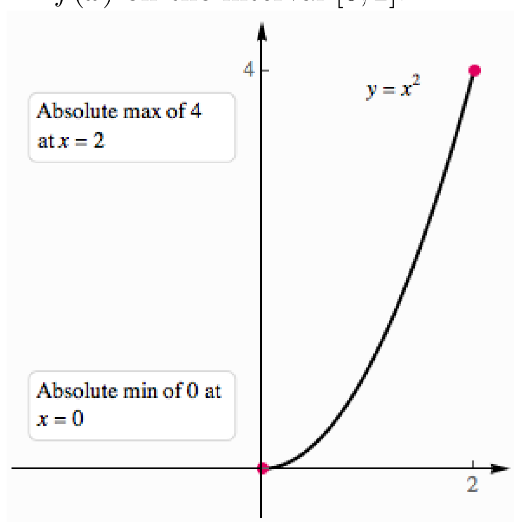
Definition 2 (Absolute Maximum and Minimum)**.** *Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an* ***absolute maximum*** *value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an* ***absolute minimum*** *value of f on D. An* ***absolute extreme value*** *is either an absolute maximum or an absolute minimum value.*

The existence and location of absolute extreme values depend on both the *function* and the *interval* of interest. The figure below shows various cases for the function $f(x) = x^2$. Note that if the interval of interest is not closed, a function might not attain absolute extreme values.

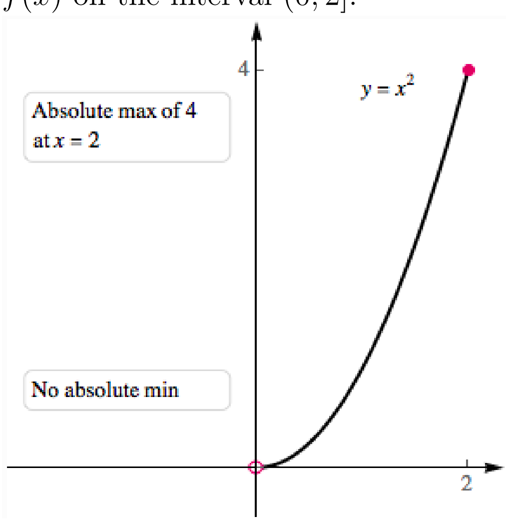
$f(x)$ on the interval $(-\infty, \infty)$:



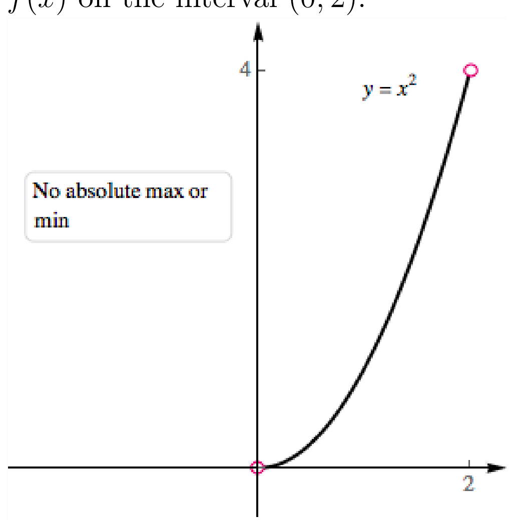
$f(x)$ on the interval $[0, 2]$:



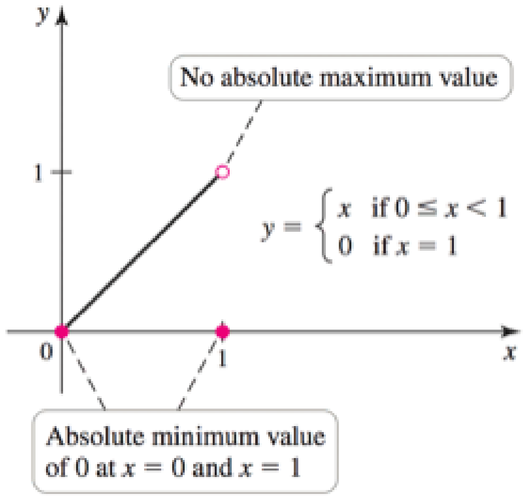
$f(x)$ on the interval $(0, 2]$:

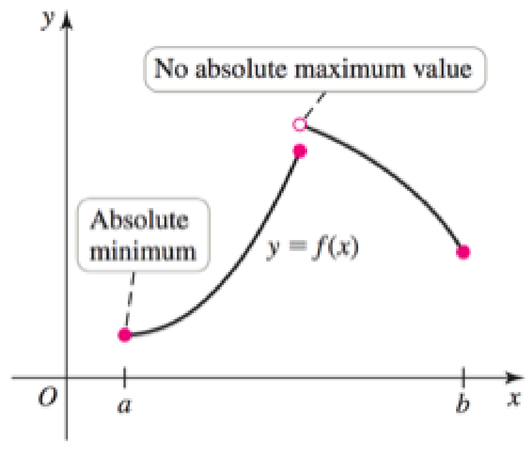


$f(x)$ on the interval $(0, 2)$:

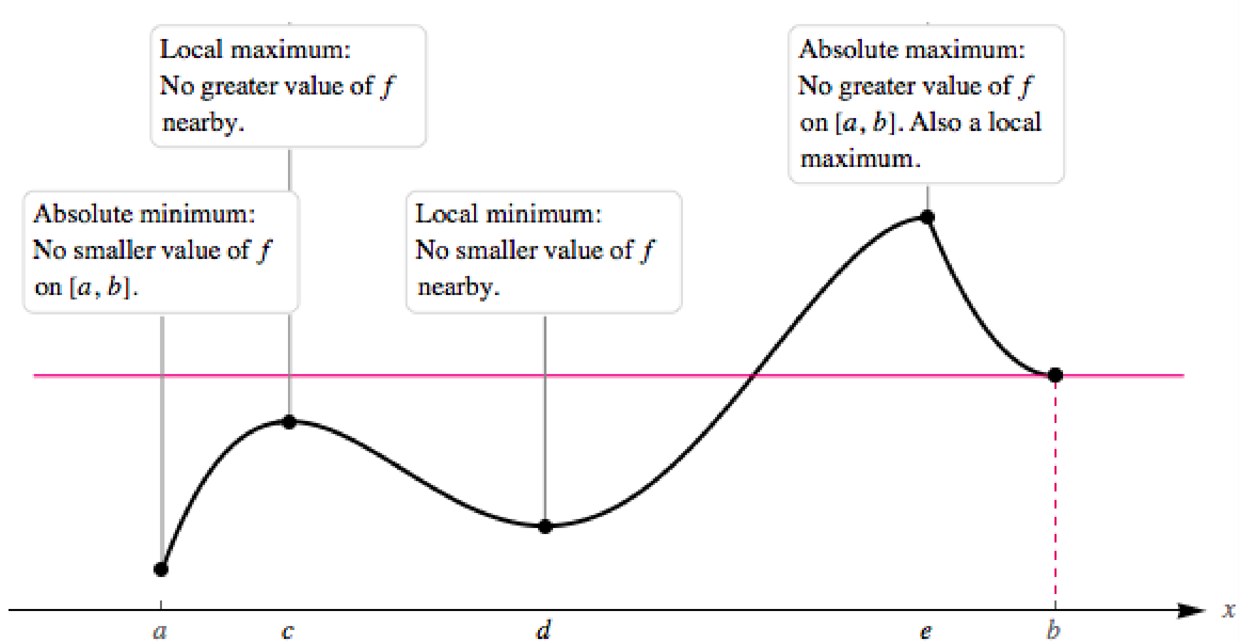


Note that defining a function on a closed interval is not enough to guarantee the existence of absolute extreme values. Both of the following functions are defined at every point of a closed interval, but neither function attains an absolute maximum—the discontinuity in each function prevents it from happening.





The function below is defined on the interval $[a, b]$. It has an absolute minimum at the endpoint a and an absolute maximum at the interior point e. In addition, the function has special behavior at c, where its value is greatest *among values at nearby points* and at d, where its value is least *among values at nearby points*. A point at which a function takes on the maximum or minimum value among values at nearby points is important.

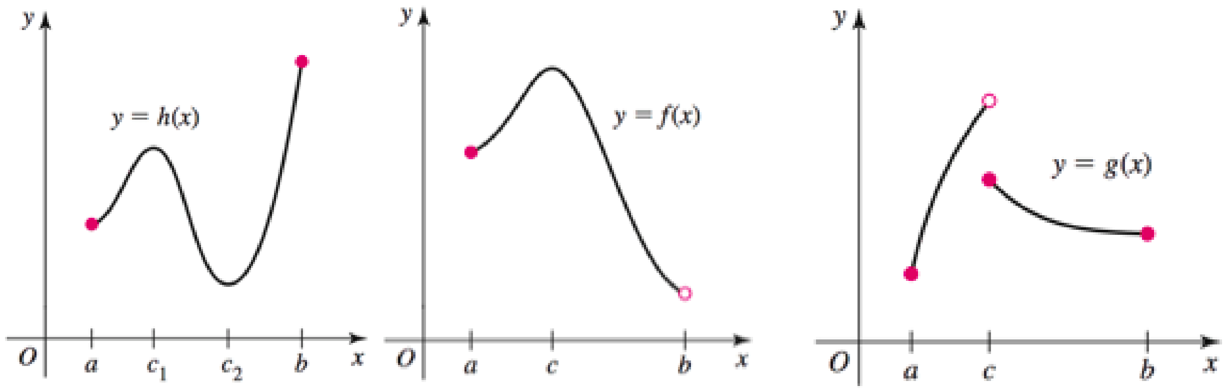


Definition 3 (Local maximum and minimum values). *Suppose c is an interior point of some interval I on which f is defined. If $f(c) \ge f(x)$ for all x in I, then $f(c)$ is a **local***

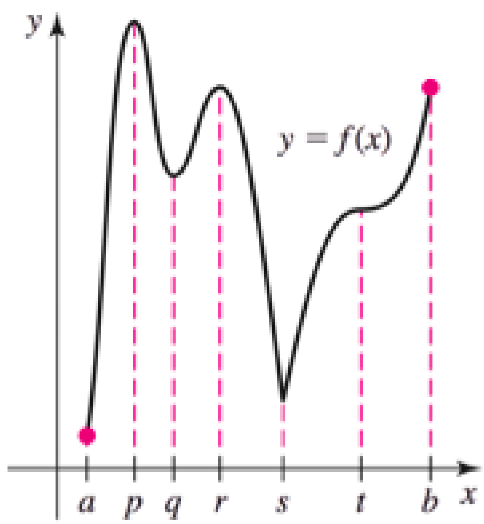
maximum *value of* f*. If* $f(c) \leq f(x)$ *for all* x *in* I*, then* $f(c)$ *is a* ***local minimum*** *value of* f*.*

> In this course, we take the convention that local maximum values and local minimum values occur only at ***interior points*** of the interval(s) of interest.

Example 4 (§4.1 Ex. 11, 12, 13)**.** *Use the following graphs to identify the points (if any) on the interval* $[a, b]$ *at which the function has an absolute maximum value or an absolute minimum value.*



Example 5 (§4.1 Ex. 16)**.** *Identify the points on the interval* $[a, b]$ *at which local and absolute extreme values occur.*



Example 6 (§4.1 Ex. 22). *Sketch a graph of a function f continuous on [0,4] satisfying the following properties: $f'(x) = 0$ at $x = 1$ and $x = 3$; $f'(2)$ is undefined; f has an absolute maximum at $x = 2$; f has neither a local maximum nor a local minimum at $x = 1$; and f has an absolute minimum at $x = 3$.*

It turns out that two conditions ensure the existence of absolute maximum and minimum values on an interval: the function must be continuous on the interval, and the interval must be closed and bounded:

Theorem 7 (Extreme Value Theorem). *A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.*

It turns out that local maxima and minima occur at points in the open interval (a, b) where the derivative is zero and at points where the derivative fails to exist. We now make this observation precise.

Theorem 8 (Local Extreme Value Theorem). *If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.*

Local extrema can also occur at points c where $f'(c)$ does not exist. Because local extrema may occur at points c where $f'(c) = 0$ and where $f'(c)$ does not exist, we make the following definition.

Definition 9 (Critical point). *An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a critical point of f.*

Example 10 (§4.1 Ex. 25, 32, 34). *Find the critical points of the following functions on the domain or on the given interval. Use a graphing utility to determine whether each critical point corresponds to a local maximum, local minimum, or neither.*

1. $f(x) = \frac{x^3}{3} - 9x$ *on* $[-7, 7]$
2. $f(x) = \sin x \cos x$ *on* $[0, 2\pi]$
3. $f(x) = x^2 - 2\ln(x^2 + 1)$
4. $f(x) = x^{2/3}$
5. $f(x) = |x - 2|$

2.1 Locating Absolute Maxima and Minima

If we have a continuous function on a closed interval **how do we locate its maximum and minimum values**?

Assume the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c_i in (a, b).
2. Evaluate f
   - at the critical points $x = c_i$
   - and at the endpoints $x = a$ and $x = b$.
3. Choose the largest and smallest values of f from Step 2.

Example 11 (§4.1 Ex. 40, 44, 48)**.** *Find the critical points of the functions below on the given intervals. Determine the absolute extreme values of f on the given interval when they exist. Use a graphing utility to confirm your conclusions.*

1. $f(x) = \dfrac{x}{(x^2+3)^2}$ *on* $[-2, 2]$

2. $f(x) = xe^{1-\frac{x}{2}}$ *on* $[0, 5]$

3. $f(x) = |2x - x^2|$ *on* $[-2, 3]$