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What is on today

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1 Maxima and minima

Briggs-Cochran-Gillett §4.1 pp. 236-245

1.1 Applications

If we have a continuous function on a closed interval, **how do we locate its absolute maximum and minimum values**?

Suppose the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c_i in (a, b).
2. Evaluate f at
   - the critical points $x = c_i$
   - and at the endpoints $x = a$ and $x = b$.
3. Choose the largest and smallest values of f from Step 2.

To locate absolute max/min on closed intervals, use the steps described above.

Example 1 (§4.1 Ex. 40, 44, 48)**.** *Find the critical points of the functions below on the given intervals. Determine the absolute extreme values of f on the given interval when they exist. Use a graphing utility to confirm your conclusions.*

1. $f(x) = \dfrac{x}{(x^2+3)^2}$ *on* $[-2, 2]$
2. $f(x) = xe^{1-\frac{x}{2}}$ *on* $[0, 5]$
3. $f(x) = |2x - x^2|$ *on* $[-2, 3]$

Example 2 (§4.1 Ex. 52)**.** *A sales analyst determines that the revenue from sales of fruit smoothies is given by*

$$R(x) = -60x^2 + 300x$$

where x is the price in dollars charged per item, for $0 \leq x \leq 5$.

a. Find the critical points of the revenue function.

b. Determine the absolute maximum value of the revenue function and the price that maximizes the revenue.

In the following example, we locate absolute max/min on an open interval by inspection of the graph of the function.

Example 3 (§4.1 Ex. 54)**.** *All rectangles with an area of 64 have a perimeter given by $P(x) = 2x + 128/x$, where x is the length of one side of the rectangle. Find the absolute minimum value for the perimeter function on the interval $(0, \infty)$. What are the dimensions of the rectangle with minimum perimeter?*

2 What derivatives tell us

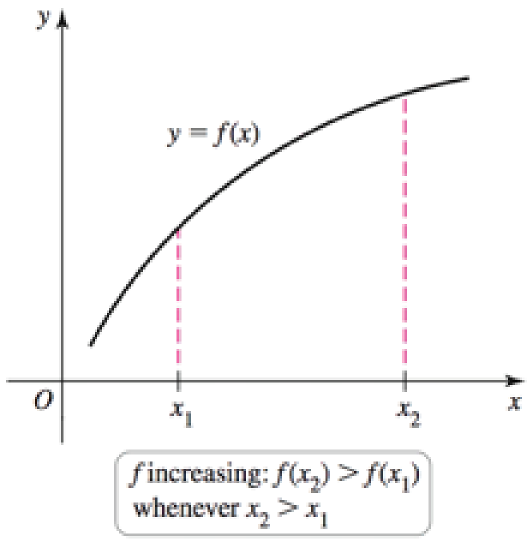
Briggs-Cochran-Gillett §4.2 pp. 245-250

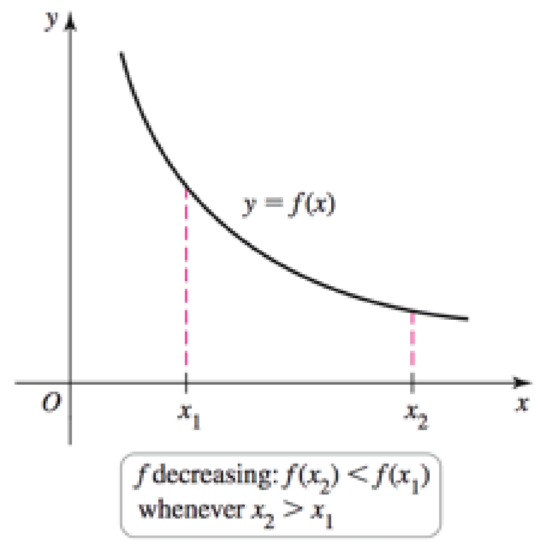
Earlier this week, we saw that the derivative helps us find critical points of functions. Today we explore how the derivative can tell us more about the behavior of functions.

2.1 Increasing and decreasing functions

Definition 4 (Increasing and decreasing functions)**.** *Suppose a function f is defined on an interval I. We say that f is* ***increasing*** *on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I*

and $x_2 > x_1$. We say that f is ***decreasing*** *on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.*





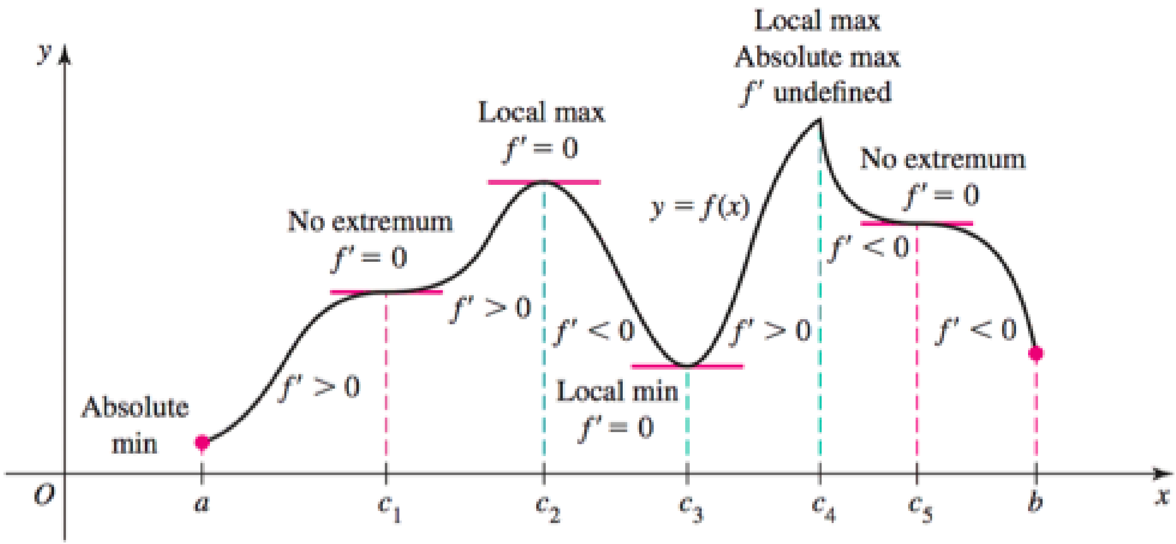
Theorem 5 (Test for intervals of increase and decrease)**.** *Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.*

Example 6 (§4.2 Ex. 18, 26, 30)**.** *For each of the following functions, find the intervals on which f is increasing and decreasing.*

1. $f(x) = x^2 - 16$
2. $f(x) = \frac{e^x}{e^{2x}+1}$
3. $f(x) = x^2\sqrt{9 - x^2}$ *on* $(-3, 3)$

2.2 Identifying local maxima and minima: The first derivative test

The figure below shows typical features of a function on an interval $[a, b]$.



At local maxima or minima (c_2, c_3, c_4), the derivative f' changes sign. Although c_1 and c_5 are critical points, the sign of f' is the same on both sides near these points, so there is no local maximum or minimum at these points. Note: *critical points do not always correspond to local extreme values*!

The observations used to interpret the figure above are summarized in a very useful test for identifying local maxima and minima:

Theorem 7 (First derivative test)**.** *Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.*

- *If f' changes sign from positive to negative as x increases through c, then f has a* ***local maximum*** *at c.*
- *If f' changes sign from negative to positive as x increases through c, then f has a* ***local minimum*** *at c.*
- *If f' is positive on both sides near c or negative on both sides near c, then f has no local extreme value at c.*

Example 8 (§4.2 Ex. 40, 45, 46)**.** *For each of the following functions:*

(A) Locate the critical points of f.

(B) Use the First Derivative Test to locate the local maximum and minimum values.

(C) Identify the absolute maximum and minimum values of the function on the given interval (when they exist).

1. $f(x) = -x^2 - x + 2$ *on* $[-4, 4]$
2. $f(x) = x^{2/3}(x-5)$ *on* $[-5, 5]$
3. $f(x) = \frac{x^2}{x^2-1}$ *on* $[-4, 4]$

Recall that the Extreme Value Theorem guarantees absolute extreme values of continuous functions on closed intervals. What can be said about absolute extrema on intervals that are not closed? The following theorem gives a helpful test:

Theorem 9 (One local extremum implies absolute extremum)**.** *Suppose f is continuous on an interval I that contains exactly one local extremum at c.*

- *If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.*
- *If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.*

Example 10 (§4.2 Ex. 50, 52)**.** *Verify that the following functions satisfy the conditions of Theorem 9 on their domains. Then find the location and value of the absolute extremum guaranteed by the theorem.*

1. $f(x) = 4x + 1/\sqrt{x}$
2. $f(x) = x\sqrt{3-x}$