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What is on today

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1 What derivatives tell us, continued

Briggs-Cochran-Gillett §4.2 pp. 245-260

1.1 First Derivative Test

Recall the First Derivative Test, discussed in the last lecture:

Theorem 1 (First Derivative Test). *Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.*

- *If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.*
- *If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.*
- *If f' is positive on both sides near c or negative on both sides near c, then f has no local extreme value at c.*

Example 2 (§4.2 Ex. 40, 45, 46). *For each of the following functions:*

(A) Locate the critical points of f.

(B) Use the First Derivative Test to locate the local maximum and minimum values.

(C) Identify the absolute maximum and minimum values of the function on the given interval (when they exist).

1. $f(x) = x^{2/3}(x-5)$ *on* $[-5, 5]$
2. $f(x) = \frac{x^2}{x^2-1}$ *on* $[-4, 4]$

Recall that the Extreme Value Theorem guarantees absolute extreme values of continuous functions on closed intervals. What can be said about absolute extrema on intervals that are not closed? The following theorem gives a helpful test:

Theorem 3 (One local extremum implies absolute extremum)**.** *Suppose f is continuous on an interval I that contains exactly one local extremum at c.*

- *If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.*
- *If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.*

Example 4 (§4.2 Ex. 50, 52)**.** *Verify that the following functions satisfy the conditions of Theorem 3 on their domains. Then find the location and value of the absolute extremum guaranteed by the theorem.*

1. $f(x) = 4x + 1/\sqrt{x}$
2. $f(x) = x\sqrt{3-x}$

1.2 Concavity and inflection points

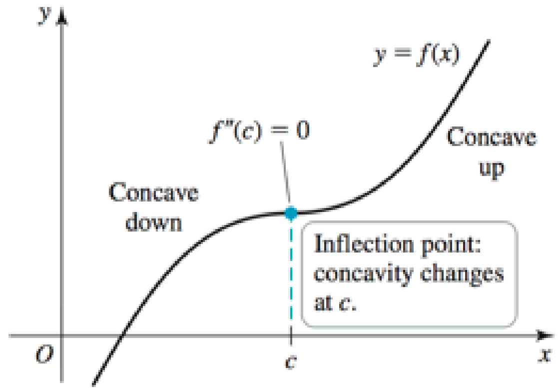
Just as the first derivative is related to the slope of tangent lines, the second derivative also has geometric meaning.

Definition 5 (Concavity and inflection point)**.** *Let f be differentiable on an open interval I. If f' is increasing on I, then f is* ***concave up*** *on I. If f' is decreasing on I, then f is* ***concave down*** *on I. If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an* ***inflection point*** *at c.*

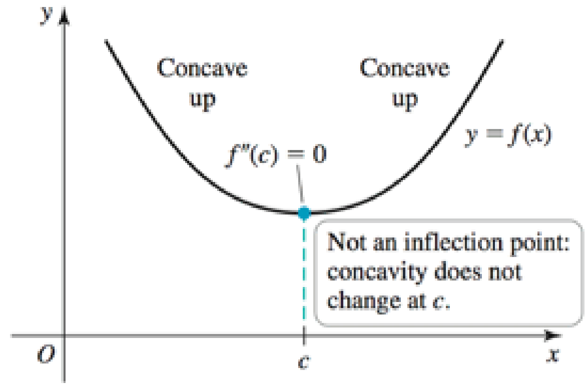
Theorem 6 (Test for concavity)**.** *Suppose that f'' exists on an open interval I.*

- *If $f'' > 0$ on I, then f is concave up on I.*
- *If $f'' < 0$ on I, then f is concave down on I.*
- *If c is a point of I at which f'' changes sign at c (from positive to negative or negative to positive), then f has an inflection point at c.*

Here are some examples:



(a)



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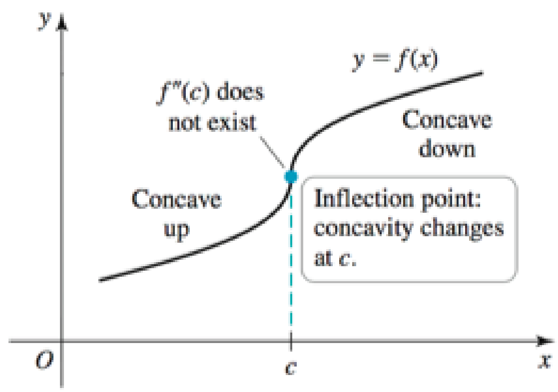
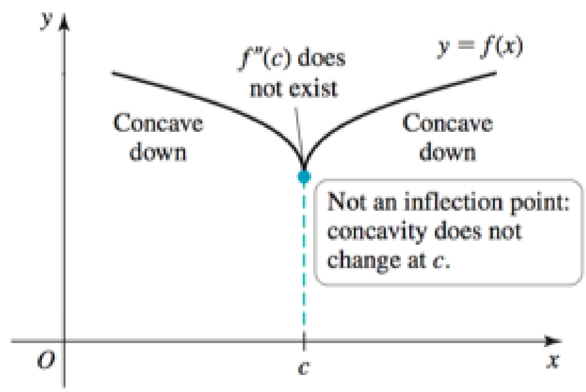


(c)



(d)

Example 7 (§4.2 Ex. 58, 62, 68)**.** *Determine the intervals on which the following functions are concave up or concave down. Identify any inflection points.*

1. $f(x) = -x^4 - 2x^3 + 12x^2$

2. $f(x) = 2x^2 \ln x - 5x^2$

3. $h(t) = 2 + \cos(2t)$ *for* $-\pi \leq t \leq \pi$.

1.3 Second Derivative Test

Theorem 8 (Second Derivative Test for local extrema)**.** *Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.*

- *If $f''(c) > 0$, then f has a local minimum at c.*
- *If $f''(c) < 0$, then f has a local maximum at c.*
- *If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.*

Example 9 (§4.2 Ex. 72, 78, 80)**.** *Locate the critical points of the following functions. Then use the Second Derivative Test to determine (if possible) whether they correspond to local maxima or local minima.*

1. $f(x) = 6x^2 - x^3$
2. $p(x) = x^4 e^{-x}$
3. $g(x) = \frac{x^4}{2-12x^2}$

Here is a recap of derivative properties: