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What is on today

1 Linear approximation

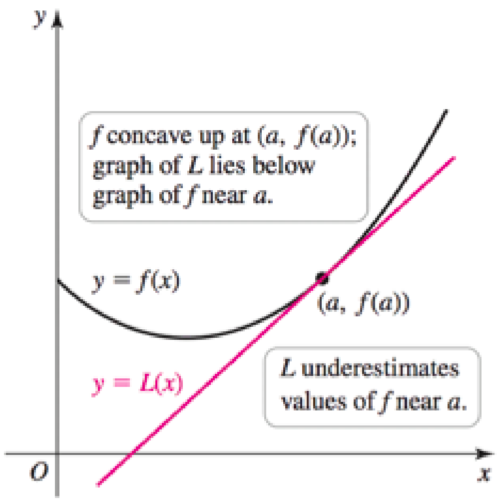
Briggs-Cochran-Gillett §4.5 pp. 281 - 287

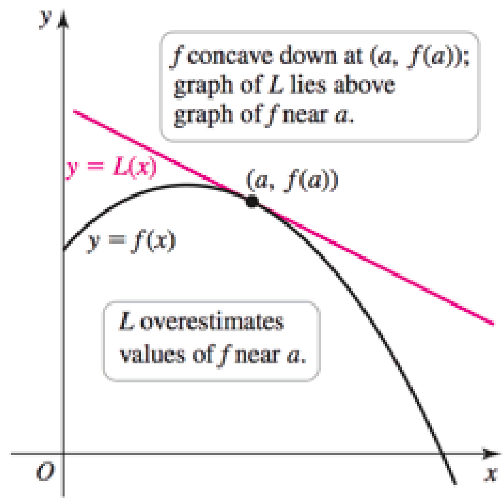
Example 1 (§4.5 Ex. 21, 24)**.** *Use linear approximations to estimate the following quantities. Choose a value of* a *that produces a small error.*

1. $1/203$
2. $\sqrt[3]{65}$

1.1 Linear approximation and concavity

We can gain a more refined understanding of what a linear approximation is telling us by studying the function's concavity. Consider the following two graphs:





In the figure on the left, f is concave up on an interval containing a, and the graph of L lies below the graph of f near a. Consequently, L is an *underestimate* of the values of f near a. In the figure on the right, f is concave down on an interval containing a. Now the graph of L lies above the graph of f, which means that the linear approximation *overestimates* the values of f near a.

Example 2 (§4.5 Ex. 32)**.** *Let* $f(x) = 5 - x^2$ *and* $a = 2$.

1. *Find the linear approximation* L *to the function* f *at the point* a.
2. *Graph* f *and* L *on the same set of axes.*
3. *Based on the graph in part (1), state whether the linear approximation to* f *near* a *is an underestimate or overestimate.*
4. *Compute* $f''(a)$ *to confirm your conclusion in part (3).*

1.2 Change in y

We can also use the linear approximation of a function to study the change in y for a corresponding change in x.

Theorem 3 (Relationship between Δx and Δy)**.** *Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a + \Delta x$ is approximately*

$$\Delta y \approx f'(a)\Delta x,$$

where $a + \Delta x$ is in I.

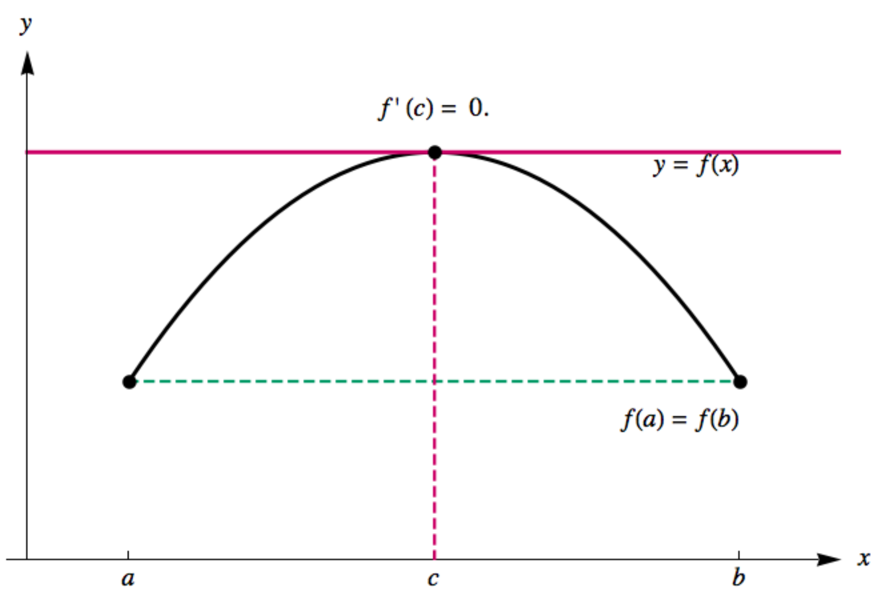
Example 4 (§4.5 Ex. 36)**.** *Approximate the change in the atmospheric pressure when the altitude increases from $z = 2$ km to $z = 2.01$ km. Use the function $P(z) = 1000e^{-z/10}$.*

2 Mean Value Theorem

Briggs-Cochran-Gillett §4.6 pp. 290 - 296

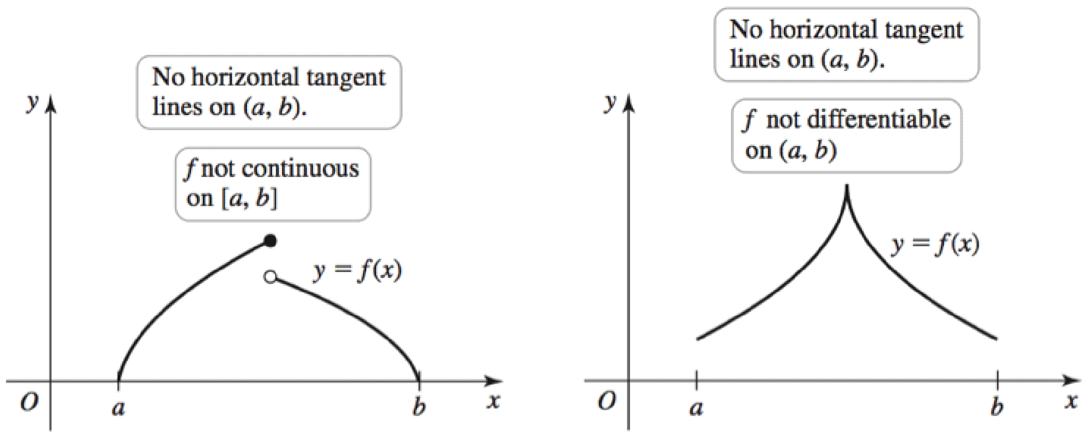
Today we will look at one of the central results in calculus: the Mean Value Theorem (MVT). Some of the theorems we have already seen about derivatives are a consequence of this theorem. We start by looking at a preliminary result, which is a particular case of the MVT: Rolle's theorem.

2.1 Rolle's Theorem



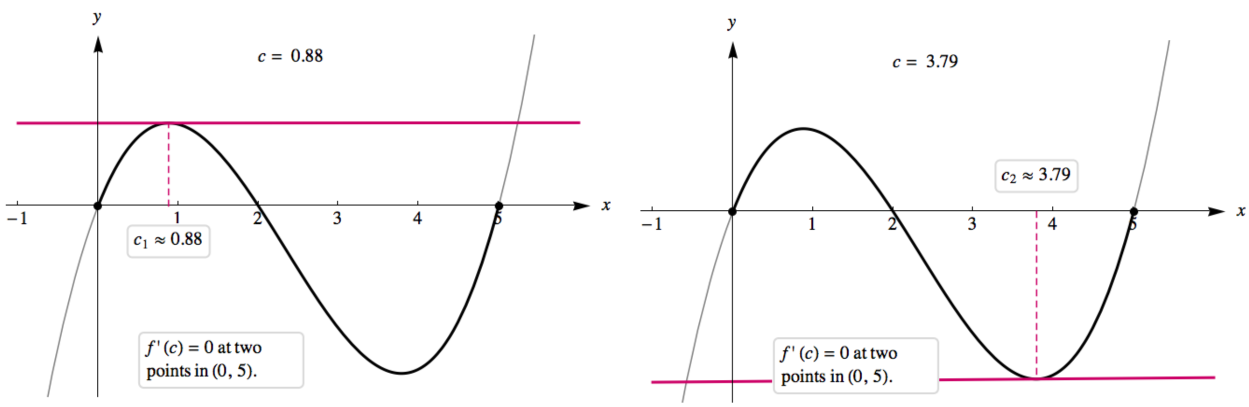
Theorem 5 (Rolle's Theorem). *Let f be a continuous function on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.*

The continuity and differentiability conditions are essential, otherwise the statement is not true!



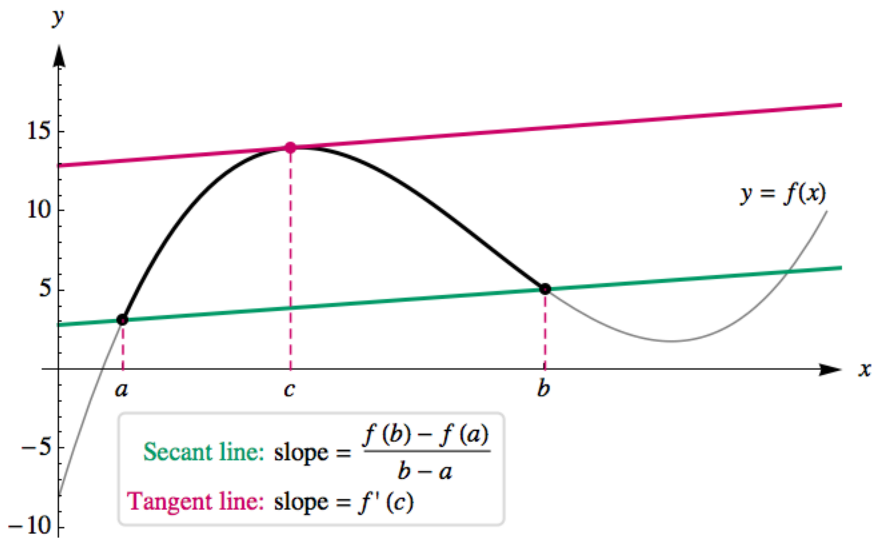
(a) (b)

Also note that there might be more than one c that satisfies the theorem:



Example 6 (§4.6 Ex. 12). *Determine whether Rolle's theorem applies to the function $f(x) = x^3 - 2x^2 - 8x$ on the interval $[-2, 4]$. If so, find the point(s) that are guaranteed to exist by Rolle's Theorem.*

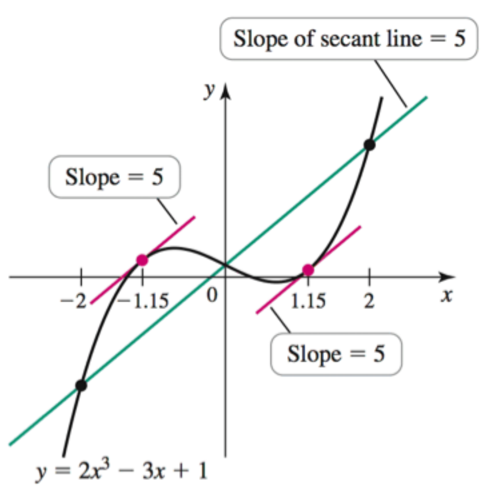
2.2 The Mean Value Theorem (MVT)'s statement



Theorem 7 (Mean Value Theorem)**.** *Let f be a continuous function on a closed interval $[a, b]$ and differentiable on (a, b). Then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note that, just like in the case of Rolle's Theorem, there might be more than one c that satisfies the MVT:



Example 8 (§4.6 Ex. 20, 23).

a. Determine whether the MVT applies to the following functions on the given interval $[a, b]$. Explain why or why not.

b. If so, find the point(s) that are guaranteed to exist by the MVT and sketch the function and the line that passes through $(a, f(a))$ and $(b, f(b))$. Mark the points P at which the slope of the function equals the slope of the secant line and sketch the tangent line at P.

1. $f(x) = \ln(2x)$, $[1, e]$
2. $f(x) = 2x^{1/3}$, $[-8, 8]$

2.3 Consequences of the MVT

Theorem 9 (Consequences of the MVT)**.**

1. ***Zero derivative implies constant function***
   If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

2. ***Functions with equal derivatives differ by a constant***
   If two functions have the property that $f'(x) = g'(x)$, for all x of an interval I, then f and g differ by a constant on the interval, i.e., $f(x) - g(x) = constant$ on I.

3. ***Intervals of increase and decrease***
   Suppose f is continuous on I and differentiable on all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

Example 10 (§4.6 Ex. 26)**.** *Without evaluating derivatives, which of the functions*

$$f(x) = \ln x,\ \ g(x) = \ln 2x,\ \ h(x) = \ln x^2 \text{ and } p(x) = \ln 10x^2$$

have the same derivative?

Example 11 (§4.6 Ex. 28)**.** *Find all functions f whose derivative is $f'(x) = x + 1$.*