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What is on today

1 L'Hôpital's Rule

Briggs-Cochran-Gillett §4.7 pp. 297 - 302

In this section, we present a new technique, *l'Hôpital's Rule*, to evaluate certain limits called *indeterminate forms.*

What is an indeterminate form? As an example, consider $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$. If we attempt to substitute $x = 0$ into $\frac{\sin x}{x}$, we get $\frac{0}{0}$. Yet we earlier showed that $\frac{\sin x}{x}$ has limit 1 at $x = 0$. This limit is an example of an indeterminate form; in particular, $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ has the indeterminate form $0/0$ because the numerator and denominator both approach 0 as $x \to 0$.

1.1 L'Hôpital's Rule for $0/0$

Theorem 1 (L'Hôpital's Rule)**.** *Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$). The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Example 2 (§4.7 Ex. 14,16,30,34)**.** *Evaluate the following limits using l'Hôpital's Rule:*

1. $\lim\limits_{x\to -1} \dfrac{x^4 + x^3 + 2x + 2}{x+1}$
2. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x^2 + 3x}$
3. $\lim\limits_{x\to \infty} \dfrac{\tan^{-1} x - \pi/2}{1/x}$

4. $\lim_{y\to 2} \dfrac{y^2+y-6}{\sqrt{8-y^2}-y}$

1.2 L'Hôpital's Rule for ∞/∞

L'Hôpital's Rule also applies directly to limits of the form $\lim_{x\to a} \dfrac{f(x)}{g(x)}$, where $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. This indeterminate form is denoted ∞/∞.

Theorem 3 (L'Hôpital's Rule for ∞/∞)**.** *Suppose that f and g are differentiable on an open interval I containing a, with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$). The rule also applies for $x \to \pm\infty, x \to a^+$, or $x \to a^-$.

Example 4 (§4.7 Ex. 38,42,44)**.** *Evaluate the following limits:*

1. $\lim_{x\to\infty} \dfrac{4x^3-2x^2+6}{\pi x^3+4}$

2. $\lim_{x\to\infty} \dfrac{\ln(3x+5e^x)}{\ln(7x+3e^{2x})}$

3. $\lim_{x\to\pi/2} \dfrac{2\tan x}{\sec^2 x}$

1.3 How to handle related indeterminate forms: $0 \cdot \infty$ and $\infty - \infty$

Here we consider two other indeterminate forms. Limits of the form $\lim_{x\to a} f(x)g(x)$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$ are denoted $0 \cdot \infty$, while limits of the form $\lim_{x\to a}(f(x) - g(x))$, where $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ are indeterminate forms we denote $\infty - \infty$. ***L'Hôpital's Rule cannot directly be applied to either of these limits! However, if we can recast these indeterminate forms into the form $0/0$ or ∞/∞, then we may apply L'Hôpital's Rule.***

Occasionally, it helps to convert a limit as $x \to \infty$ to a limit as $t \to 0^+$ (or vice versa) by a change of variables. To evaluate $\lim_{x\to\infty} f(x)$, we define $t = \frac{1}{x}$ and note that as $x \to \infty$, we have $t \to 0^+$. Then $\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f\left(\frac{1}{t}\right)$.

Example 5 (§4.7 Ex. 46, 49, 51, 52)**.** *Evaluate the following limits:*

1. $\lim_{x\to 1^-} (1-x)\tan\frac{\pi x}{2}$
2. $\lim_{x\to \pi/2^-} \left(\frac{\pi}{2} - x\right)\sec x$
3. $\lim_{x\to 0^+} \left(\cot x - \frac{1}{x}\right)$
4. $\lim_{x\to\infty} \left(x - \sqrt{x^2+1}\right)$

1.4 Indeterminate forms 1^∞, 0^0 and ∞^0

The indeterminate forms $1^\infty,\ 0^0,\ \infty^0$ arise from limits of the form

$$\lim_{x\to a,\pm\infty,a^+,a^-} f(x)^{g(x)}.$$

L'Hôpital's rule cannot be applied directly, but we can rewrite the expression inside the limit as an exponential:

$$f(x)^{g(x)} = e^{g(x)\ln(f(x))}.$$

Using this and the fact that the exponential function is continuous we know that

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln(f(x))} = e^{\lim_{x\to a} g(x)\ln(f(x))}.$$

So we just need to worry about the limit $\lim_{x\to a} g(x)\ln(f(x))$. Let us illustrate this in some examples.

Example 6 (§4.7 Ex. 56, 62, 65, 66)**.** *Evaluate the following limits or explain why they do not exist:*

1. $\lim_{x\to 0}(1+4x)^{3/x}$
2. $\lim_{x\to 0}(e^{5x}+x)^{1/x}$
3. $\lim_{x\to 0^+}(\tan x)^x$
4. $\lim_{z\to\infty}\left(1+\dfrac{10}{z^2}\right)^{z^2}$

So all indeterminate forms are (l'Hôpital's rule just applies directly to the 2 first forms):

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 1^{\infty},\ 0^0,\ \infty^0.$$

Limits that are **NOT** indeterminate forms:

- $1/\infty$
- $1/0$
- $0/\infty$
- $\infty/0$
- 0^{∞}