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What is on today

1 L'Hôpital's Rule: Part 2

Briggs-Cochran-Gillett §4.7 pp. 302 - 307

1.1 Growth rates

Our goal now is to use what we know about limits, including l'Hôpital's rule, to obtain *a ranking* of the functions we know **based on their growth rates**.

Definition 1 (Growth Rates of functions as $x \to \infty$)**.** *Let f and g be functions such that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then*

- *f* ***grows faster than*** *g as $x \to \infty$ is*

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or, equivatently, } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

  We write $g(x) << f(x)$.

- *f and g* ***have comparable growth rates*** *if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

  with $M > 0$.

We have:

$$\ln x << x^p << e^x, \ p > 0.$$

More generally

$$\ln^q(x) << x^p << x^p \ln^r x << e^x,$$

where $q, p, r > 0$.

Example 2 (§4.7 Ex. 70, 73, 72, 80)**.** *Use limits to determine which of the following functions grows faster or state they have comparable growth rates:*

1. $x^2 \ln x$ *and* $\ln^2 x$
2. 100^x *and* x^x
3. $\ln x$ *and* $\ln(\ln x)$
4. e^{x^2} *and* $x^{x/10}$

2 Limits Wrap-up

Example 3 (§4.7 Ex. 86, 88, 96, 97)**.** *Use any method to evaluate the following limits:*

1. $\lim\limits_{x\to\infty} x^2 \ln(\cos(1/x))$
2. $\lim\limits_{x\to\pi/2} (\pi - 2x)\tan x$
3. $\lim\limits_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$
4. $\lim\limits_{x\to 1} \frac{x\ln x - x + 1}{x\ln^2 x}$

3 Antiderivatives

Briggs-Cochran-Gillett §4.9 pp. 319 - 324

3.1 Definition and first examples

The reverse process to differentiation is called **antidifferentiation**.

Definition 4. *A function F is an **antiderivative** of f (on an interval I) if $F'(x) = f(x)$ (on I).*

For example

$$\frac{d}{dx}x = 1$$

hence $F(x) = x$ is an antiderivative of $f(x) = 1$. Or

$$\frac{d}{dx}e^x = e^x$$

hence $F(x) = e^x$ is an antiderivative of $f(x) = e^x$. Of course we also have

$$\frac{d}{dx}(e^x + 2) = e^x$$

so $F_1(x) = e^x + 2$ is another antiderivative of $f(x) = e^x$. Actually we have seen that one of the consequences of the MVT is that

a function $f(x)$ differs from another function $g(x)$ by a constant
if and only if $f'(x) = g'(x)$.

This proves the following theorem:

Theorem 5. *Let F be an antiderivative of f on an interval I. Then all the antiderivatives of f on I have the form $F(x) + C$ where C is an arbitrary constant.*

*We write $\int f(x)dx = F(x) + C$ and say that $F(x) + C$ is the **indefinite integral** of f.*

Example 6 (§4.9 Ex. 12, 14, (based on) 19, 20)**.** *Find all antiderivatives of the following functions. Check your work by taking derivatives.*

1. $g(x) = 11x^{10}$

2. $g(x) = -4\cos(4x)$

3. $f(x) = 2e^{2x}$

4. $h(y) = y^{-1}$

3.2 Indefinite integrals to know

It is useful to know some of the more common indefinite integrals:

- $\displaystyle\int K dx = Kx + C$
- $\displaystyle\int x^p dx = \frac{x^{p+1}}{p+1} + C$, if $p \neq -1$
- $\displaystyle\int \frac{1}{x} dx = \ln|x| + C$
- $\displaystyle\int \cos(ax) dx = \frac{1}{a}\sin(ax) + C$
- $\displaystyle\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + C$
- $\displaystyle\int \sec^2(ax) dx = \frac{1}{a}\tan(ax) + C$
- $\displaystyle\int \sec(ax)\tan(ax) dx = \frac{1}{a}\sec(ax) + C$
- $\displaystyle\int e^{ax} dx = \frac{1}{a}e^{ax} + C$
- $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
- $\displaystyle\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
- $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\left|\frac{x}{a}\right|\right) + C$

Also, by the corresponding rules for differentiation, we have that

- $\displaystyle\int cf(x) dx = c\int f(x) dx$
- $\displaystyle\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx$

Example 7 (§4.9 Ex. 24, 31, 38, 50, 57, 58, 40)**.** *Determine the following indefinite integrals. Check your work by taking derivatives.*

1. $\int (3u^{-2} - 4u^2 + 1)du$

2. $\int (3x + 1)(4 - x)dx$

3. $\int (\sin(4t) - \sin(t/4))\, dt$

4. $\int \frac{3}{4 + v^2}dv$

5. $\int e^{x+2}dx$

6. $\int \frac{10t^5 - 3}{t}dt$

7. $\int 2\sec^2(2v)dv$