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What is on today

1 Antiderivatives wrap-up

Briggs-Cochran-Gillett §4.9 pp. 319 - 324

Example 1 (§4.9 Ex. 24, 31, 38, 50, 57, 58, 40)**.** *Determine the following indefinite integrals. Check your work by taking derivatives.*

1. $\int (3u^{-2} - 4u^2 + 1)du$
2. $\int (3x + 1)(4 - x)dx$
3. $\int (\sin(4t) - \sin(t/4))\, dt$
4. $\int \frac{3}{4 + v^2} dv$
5. $\int e^{x+2} dx$
6. $\int \frac{10t^5 - 3}{t} dt$
7. $\int 2\sec^2(2v)dv$

2 Introduction to differential equations

Briggs-Cochran-Gillett §4.9 pp. 325 - 329

2.1 Initial value problems

An equation involving an unknown function and its derivatives is called a *differential equation*. For example, suppose you know that the derivative of a function f satisfies the equation

$$f'(x) = 2x + 10.$$

To find a function f that satisfies this equation, we note that the solutions are antiderivatives of $2x + 10$, which are

$$f(x) = x^2 + 10x + C,$$

where C is an arbitrary constant.

Now suppose we further gave an *initial condition* of

$$f(1) = 13;$$

this then would allow us to determine the constant. By substituting in 1, we find

$$f(1) = 1^2 + 10 \cdot 1 + C = 13$$

which implies that $C = 2$. This then gives the solution that

$$f(x) = x^2 + 10x + 2.$$

A differential equation together with an initial condition is called an *initial value problem.*

Example 2 (§4.9 Ex. 62, 66)**.** *For the following functions f, find the antiderivative F that satisfies the given condition.*

1. $f(x) = \frac{4\sqrt{x}+6/\sqrt{x}}{x^2}$; $F(1) = 4$

2. $f(\theta) = 2\sin(2\theta) - 4\cos(4\theta)$; $F\left(\frac{\pi}{4}\right) = 2$

Example 3 (§4.9 Ex. 73, 74)**.** *Find the solution of the following initial value problems.*

1. $y'(t) = \frac{3}{t} + 6; y(1) = 8$

2. $u'(x) = \frac{e^{2x}+4e^{-x}}{e^x}; u(\ln 2) = 2$

2.2 One-dimensional motion

Antiderivatives allow us to revisit the topic of one-dimensional motion. Suppose the position of an object that moves along a line relative to an origin is $s(t)$, where $t \geq 0$ measures elapsed time. The velocity of the object is $v(t) = s'(t)$, which we now re-interpret in terms of antiderivatives: *The position function is an antiderivative of the velocity.* If we are given the velocity function of an object and its position at a particular time, we can determine its position at all future times by solving an initial value problem.

Moreover, we know that the acceleration $a(t)$ of an object moving in one dimension satisfies $a(t) = v'(t)$. This says that velocity is an antiderivative of the acceleration. So if we are given the acceleration of an object and its velocity at a particular time, we can determine its velocity at all times. To summarize:

Theorem 4 (Initial value problems for velocity and position)**.** *Suppose an object moves along a line with a velocity $v(t)$ for $t \geq 0$. Then its position is found by solving the initial value problem*

$$s'(t) = v(t), s(0) = s_0, \quad \textit{where } s_0 \textit{ is the initial position.}$$

If the acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v'(t) = a(t), v(0) = v_0, \quad \textit{where } v_0 \textit{ is the initial velocity.}$$

Example 5 (§4.9 Ex. 84)**.** *Given the velocity function $v(t) = e^{-2t} + 4$ of an object moving along a line, find the position function with the given initial position $s(0) = 2$. Then graph both the velocity and position function.*

Example 6 (§4.9 Ex. 90)**.** *Given the acceleration function $a(t) = 4$ of an object moving along a line, find the position function with the following given initial velocity and position: $v(0) = -3, s(0) = 2$.*

Example 7 (§4.9 Ex. 98)**.** *Consider the following description of the vertical motion of an object subject only to the acceleration due to gravity: A stone is thrown vertically upward with a velocity of 30 m/s from the edge of a cliff 200 m above a river.*

Begin with the acceleration equation $a(t) = v'(t) = g$, where $g = -9.8m/s^2$.

1. *Find the velocity of the object for all relevant times.*
2. *Find the position of the object for all relevant times.*
3. *Find the time when the object reaches its highest point. What is the height?*
4. *Find the time when the object strikes the ground.*

3 Approximating area under curves and Riemann sums

Briggs-Cochran-Gillett §5.1 pp. 333 - 339

3.1 Introduction

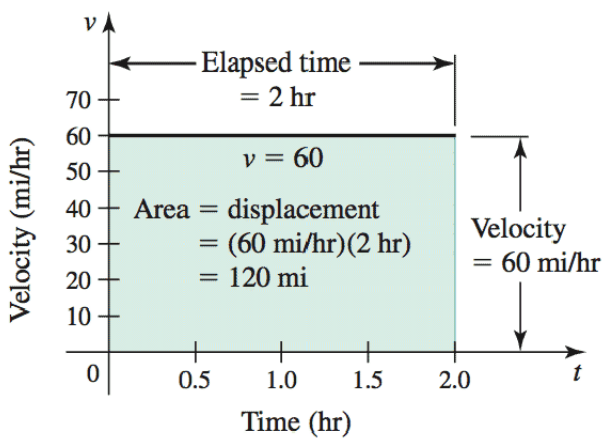
Geometrically, we know that if we graph the displacement function $s(t)$, the derivative function—which gives the velocity $v(t)$ at each instant—gives the slope of the tangent to each point of the graph of $s(t)$.

We also know that if we start with the velocity function $v(t)$, we can compute its antiderivative to find $s(t)$. What can we say about this geometrically? If we have the graph of $v(t)$, how can we find $s(t)$? This is an important question that we are going to explore during the next few classes.

If the velocity is constant, the answer is simple, and it points us in the right direction: for example, consider a car moving in a line at constant velocity of 60 mi/hr. Then the displacement is simply the velocity times the time. If the car drove for two hours, then

$$s(2) = 60 \times 2 = 120\ mi.$$

Geometrically, this is the area under the graph of the velocity:

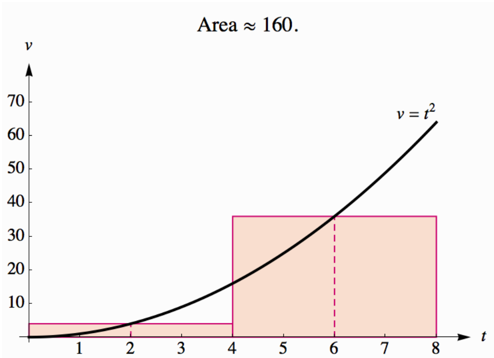


If the velocity is not constant, the displacement is not simply velocity times time. But what if we subdivided the time into intervals where we could approximate the velocity as being constant? Then we would be able to calculate and approximate displacement!

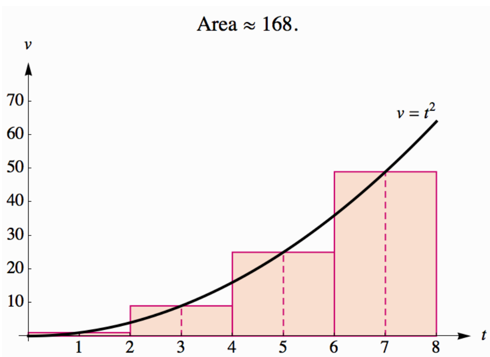
3.2 Approximating area under curves

Consider a car moving with velocity $v(t) = t^2\ mi/hr$, from $t = 0$ to $t = 8$.

- Subdivision into 2 intervals:



- Subdivision into 4 intervals:



To get better approximations, we could continue to subdivide the interval: see interactive Figure 5.5, Chapter 5.1 of the textbook.

In this case we used the **midpoint** value for an approximate value of the velocity on the interval. We could also have used the **left endpoint** or the **right endpoint** (or any other point really...)

Example 8 (§5.1, Ex. 11)**.** *If the velocity of an object is* $v(t) = 2t + 1$ (m/s)*, approximate the displacement of the object on* $0 \leq t \leq 8$ *by subdividing the interval in 2 subintervals.* ***Use the left endpoint of each subinterval to compute the height of the rectangles****.*