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# What is on today



# 1 Approximating area under curves and Riemann sums

Briggs-Cochran-Gillett §5.1 pp. 333 - 347

## 1.1 Riemann sums

The process of **approximating the area below a curve** using areas of rectangles is known as computing **Riemann sums**. This is useful for much more than computing displacements and deserves our attention.

**Definition 1** (Regular Partition)**.** *Given an interval* $[a, b]$ *and a positive integer* $n$*, we compute*

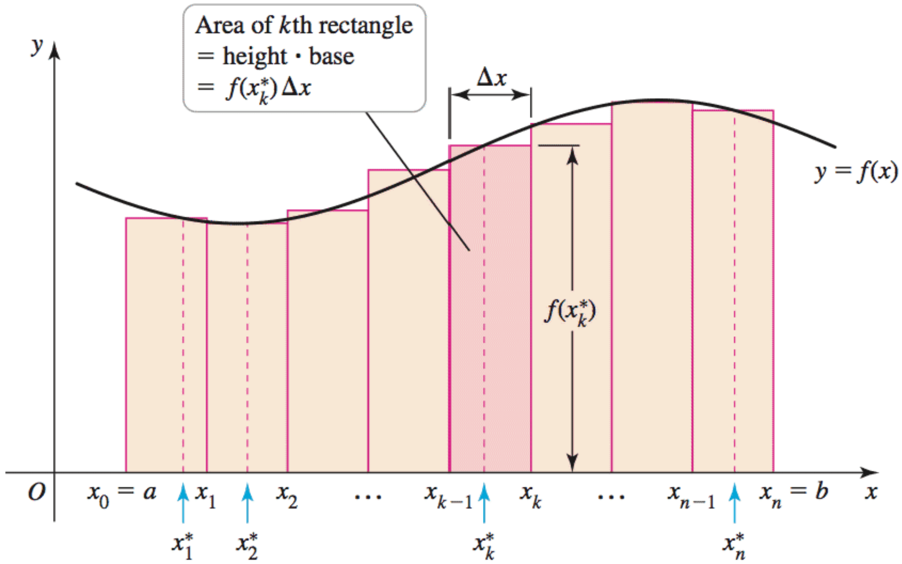
$$\Delta x = \frac{b-a}{n}.$$

*Then we subdivide the interval into* $n$ *subintervals by letting*

$$x_0 = a \text{ and } x_k = x_{k-1} + \Delta x.$$

*Then* $x_n = b$*. The endpoints* $x_0, x_1, \ldots, x_n$ *of the subintervals are called* ***grid points****. The equally spaced numbers* $x_k$ *form a* ***regular partition*** *of* $[a, b]$*.*

Let $f$ be a function defined on an interval $[a, b]$ that we have partitioned into $n$ intervals.

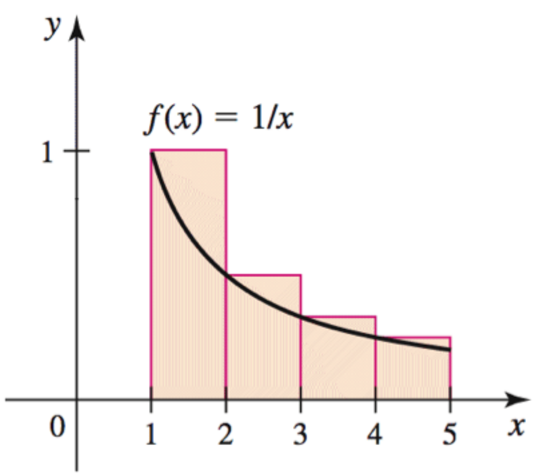


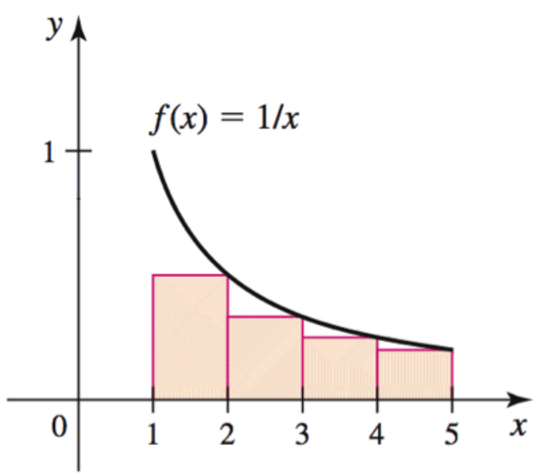

A **Riemann sum** is computed by adding the areas of any rectangles with bases in the subintervals in the partition and height equal to $f(x_k^*)$ where $x_k^*$ is some point in the interval:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x.$$

- If $x_k^*$ is the left endpoint of $[x_{k-1}, x_k]$ then we call it a **left Riemann sum**
- If $x_k^*$ is the right endpoint of $[x_{k-1}, x_k]$ then we call it a **right Riemann sum**
- If $x_k^*$ is the midpoint of $[x_{k-1}, x_k]$ then we call it a **midpoint Riemann sum**

**Example 2** (§5.1, Ex. 18)**.** *Calculate the left and right Riemann sums for* $f(x) = 1/x$ *on* $[1, 5]$*, with* $n = 4$*.*



**Example 3** (§5.1, Ex. 34). *Let $f(x) = 4 - x$ on $[-1, 4]$, $n = 5$.*

1. *Sketch the graph of the function on the given interval.*
2. *Calculate $\Delta x$ and the grid points $x_0, \ldots, x_n$.*
3. *Illustrate the midpoint Riemann sum by sketching the appropriate rectangles.*
4. *Calculate the midpoint Riemann sum.*

## 1.2 Area under the velocity curve

When we approximate areas under curves using Riemann sums, we can incrementally subdivide the interval into smaller and smaller pieces. This is a very important idea, and our first result about it is the following:

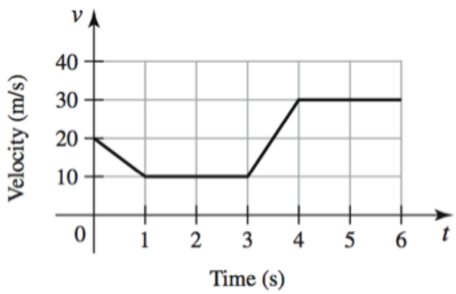
**Theorem 4.** *If $f$ is a positive continuous function on $[a, b]$ then if we take smaller and smaller partitions of $[a, b]$, the Riemann sums are converging to a number that is the area under the curve between $x = a$ and $x = b$.*

Going back to our original example, when we approximate the displacement by the approximate area under the velocity graph, if we take smaller and smaller rectangles we get better and better approximations. By the theorem above, if we take the limit of this process, our sums converge to the area under the graph that would be the precise displacement for the relevant interval of time. Hence

$$\text{If the velocity is positive, the area under velocity graph between } t_0 \text{ and } t_1 \\ = \\ \text{the displacement between } t_0 \text{ and } t_1.$$

Let's see this in an illustration:

**Example 5** (§5.1, Ex. 66)**.** *Consider the velocity of an object moving along a line:*




1. *Describe the motion of the particle over the interval* $[0, 6]$.
2. *Use geometry to find the displacement of the object between* $t = 0$ *and* $t = 3$.
3. *Use geometry to find the displacement of the object between* $t = 3$ *and* $t = 5$.
4. *Assuming that the velocity remains* 30 *m/s for* $t \geq 4$, *find the function that gives the displacement between* $t = 0$ *and any* $t \geq 5$.

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When working with Riemann sums, *sigma notation* can be used to express these sums in a compact way.

## 1.3 Sigma notation

For example, the sum

$$1 + 2 + 3 + \cdots + 10$$

is written in sigma notation as

$$\sum_{k=1}^{10} k.$$

Here are two useful properties of sigma noation:

- Constant multiple rule: Let $c$ be a constant. Then $\displaystyle\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$.
- Addition rule: $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.

## 1.4 Sums of powers

The following formulas for sums of powers of integers are also very useful:

**Theorem 6** (Sums of powers of integers)**.** *Let $n$ be a positive integer and $c$ a real number.*

1. $\displaystyle\sum_{k=1}^{n} c = cn$

2. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

3. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

**Example 7** (§5.1 Ex. 41 a, b, g, h)**.** *Evaluate the following expressions:*

1. $\displaystyle\sum_{k=1}^{10} k$

2. $\displaystyle\sum_{k=1}^{6} (2k+1)$

3. $\displaystyle\sum_{p=1}^{5} (2p+p^2)$

4. $\displaystyle\sum_{n=0}^{4} \sin\frac{n\pi}{2}$

### 1.5 Riemann sums using sigma notation

With sigma notation, a Riemann sum has the convenient compact form

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

We can use this to rewrite left, right, and midpoint Riemann sums:

**Definition 8** (Left, right, and midpoint Riemann sums in sigma notation)**.** *Suppose $f$ is defined on an interval $[a,b]$, which is divided into $n$ subintervals of equal length $\Delta x$. If $x_k^*$ is a point in the $k$th subinterval $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$, then the Riemann sum for $f$ on $[a,b]$ is $\sum_{k=1}^{n} f(x_k^*)\Delta x$. Here are our three cases:*

1. *Left Riemann sum:* $x_k^* = a + (k-1)\Delta x$
2. *Right Riemann sum:* $x_k^* = a + k\Delta x$
3. *Midpoint Riemann sum:* $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$

**Example 9** (§5.1 Ex. 44)**.** *Let $f(x) = x^2 + 1$ on $[-1, 1]$. Letting $n = 50$, use sigma notation to write the left, right, and midpoint Riemann sums.*

## 2 Definite integrals

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| Briggs-Cochran-Gillett §5.2 pp. 348 - 352 |

### 2.1 Net area

So far, we have been considering functions $f$ which are nonnegative on an interval $[a,b]$. Now we will discover the geometric meaning of Riemann sums when $f$ is negative on some or all of $[a,b]$.

Consider the function $f(x) = 1 - x^2$ on the interval $[1,3]$ with $n = 4$. We compute a midpoint Riemann sum. The length of each subinterval is $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$. The grid points are

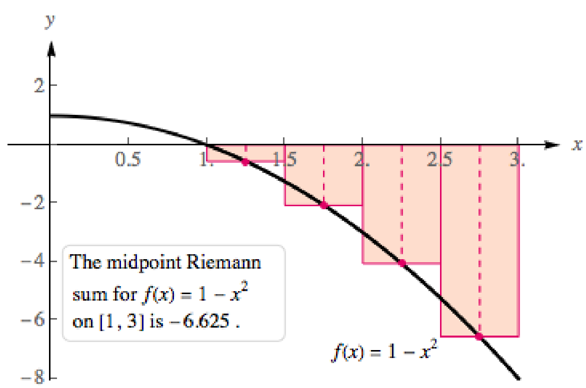
$$x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3.$$

We compute the midpoints of the subintervals:

$$x_1^* = 1.25, x_2^* = 1.75, x_3^* = 2.25, x_4^* = 2.75.$$

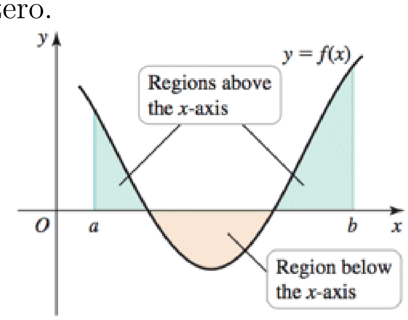
So the midpoint Riemann sum is

$$\sum_{k=1}^{4} f(x_k^*)(0.5) = (f(1.25) + f(1.75) + f(2.25) + f(2.75))(0.5) = -6.625.$$




Note that all values of $f(x_k^*)$ are negative, so the Riemann sum is also negative. Indeed, the Riemann sum is an approximation to the *negative* of the area of the region bounded by the curve and the $x$-axis.

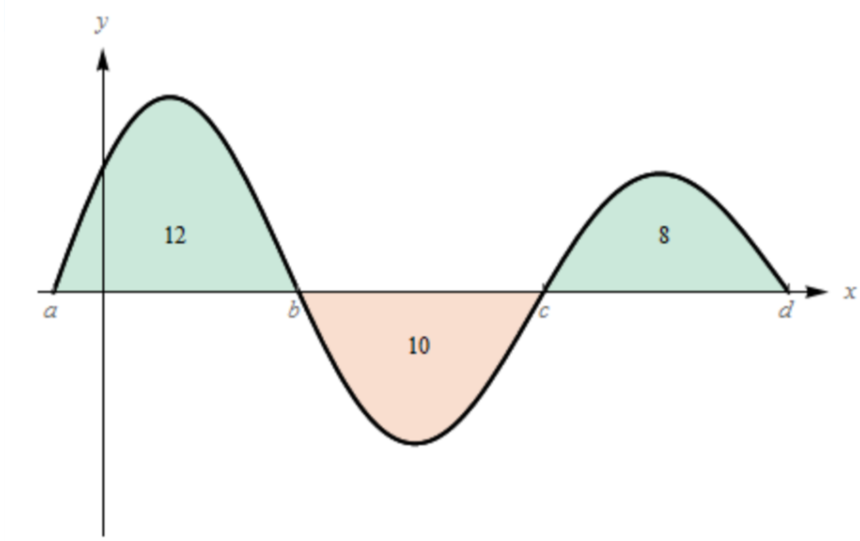
More generally, if $f$ is positive on only part of $[a, b]$, we get positive contributions to the Riemann sum where $f$ is positive and negative contributions to the Riemann sum where $f$ is negative. In this case, Riemann sums approximate the area of the regions that lie above the $x$-axis minus the area of the regions that lie below the $x$-axis. This difference between the positive and negative contributions is called the ***net area***. It can be positive, negative, or zero.




**Definition 10** (Net area)**.** *Consider the region $R$ bounded by the graph of a continuous function $f$ and the $x$-axis between $x = a$ and $x = b$. The net area of $R$ is the sum of the areas of the parts of $R$ that lie above the $x$-axis minus the area of the parts of $R$ that lie below the $x$-axis on $[a, b]$.*

**Geometrically:**

The *definite integral* corresponds to the net area:




We have $\int_a^b f(x)dx = 12$, $\int_b^c f(x)dx = -10$, $\int_a^d f(x)dx = 10$.

Below is a formal definition.

## 2.2 Definition of definite integral

Riemann sums for $f$ on $[a, b]$ approximate the net area of the region bounded by the graph of $f$ and the $x$-axis between $x = a$ and $x = b$. How do we make these approximations exact? If $f$ is continuous on $[a, b]$, it is reasonable to expect the Riemann sum approximations to approach the exact value of the net area as the number of subintervals $n \to \infty$ and as the length of the subintervals $\Delta x \to 0$, giving net area $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$. This brings us to the notion of the definite integral:

**Definition 11** (Definite integral)**.** *A function $f$ defined on $[a, b]$ is integrable on $[a, b]$ if the limit $\lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists. This limit is the* ***definite integral of** $f$ **from** $a$ **to** $b$, which we write*

$$\int_a^b f(x)dx = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$