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What is on today

1 Definite integrals

Briggs-Cochran-Gillett §5.2 pp. 352-361

1.1 From Riemann sums to definite integrals

Example 1 (§5.2 Ex. 15, 16)**.** *The following functions are positive and negative on the given interval.*

(a) Sketch the function on the given interval.

(b) Approximate the net area bounded by the graph of f and the x-axis on the interval using a left, right, and midpoint Riemann sum with $n = 4$.

1. $f(x) = 4 - 2x$ *on* $[0, 4]$

2. $f(x) = 8 - 2x^2$ *on* $[0, 4]$

1.2 Definition of definite integral

Riemann sums for f on $[a, b]$ approximate the net area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$. How do we make these approximations exact? If f is continuous on $[a, b]$, it is reasonable to expect the Riemann sum approximations to approach the exact value of the net area as the number of subintervals $n \to \infty$ and as the length of the subintervals $\Delta x \to 0$, giving net area $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$. This brings us to the notion of the definite integral:

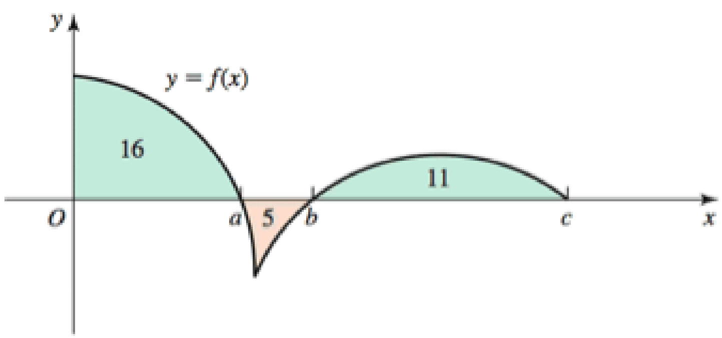
Definition 2 (Definite integral)**.** *A function f defined on $[a, b]$ is integrable on $[a, b]$ if the limit $\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists. This limit is the **definite integral of** f **from** a **to** b, which we write*

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Example 3 (§5.2 Ex. 22)**.** *Consider the limit $\lim_{\Delta x \to 0} \sum_{k=1}^{n} (4 - x_k^{*2})\Delta x_k$ on $[-2, 2]$ of Riemann sums for a function f on $[a, b]$. Identify f and express the limit as a definite integral.*

1.3 Evaluating definite integrals

Example 4 (§5.2 Ex. 33, 34, 35, 36)**.** *The figure shows the areas of regions bounded by the graph of f and the x-axis. Evaluate the following integrals.*



1. $\int_0^a f(x)dx$
2. $\int_0^b f(x)dx$
3. $\int_a^c f(x)dx$
4. $\int_0^c f(x)dx$

We can use familiar area formulas from geometry to evaluate certain definite integrals.

Example 5 (§5.2 Ex. 30)**.** *Use geometry (not Riemann sums) to evaluate the definite integral*

$$\int_{-1}^{3} \sqrt{4-(x-1)^2}dx.$$

Sketch a graph of the integrand, show the region in question, and interpret your result.

We can also write down Riemann sums, take the limit as $n \to \infty$, and use the formulas for sums of powers of integers to compute certain definite integrals.

Example 6 (§5.2 Ex. 48, 50)**.** *Use the definition of the definite integral to evaluate the following definite integrals. Use right Riemann sums and results on sums of powers of integers.*

1. $\int_{1}^{5} (1-x)dx$

2. $\int_{0}^{2} (x^2-1)dx$

1.4 Properties of definite integrals

We first establish some criteria for a function to be integrable:

Theorem 7 (Integrable functions)**.** *If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.*

Here are some very important properties of definite integrals:

Let f and g be integrable functions on an interval that contains a, b, and p.

1. $\int_a^a f(x)\,dx = 0$ Definition

2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition

3. $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

4. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$ For any constant c

5. $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$

6. The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

Example 8 (§5.2 Ex. 42)**.** *Suppose $\int_1^4 f(x)dx = 8$ and $\int_1^6 f(x)dx = 5$. Evaluate the following integrals.*

1. $\int_1^4 (-3f(x))dx$
2. $\int_1^4 3f(x)dx$
3. $\int_6^4 12f(x)dx$
4. $\int_4^6 3f(x)dx$

Example 9 (§5.2 Ex. 44)**.** *Suppose $f(x) \geq 0$ on $[0, 2]$, $f(x) \leq 0$ on $[2, 5]$, $\int_0^2 f(x)dx = 6$, and $\int_2^5 f(x)dx = -8$. Evaluate the following integrals.*

1. $\int_0^5 f(x)dx$
2. $\int_0^5 |f(x)|dx$
3. $\int_2^5 4|f(x)|dx$
4. $\int_0^5 (f(x) + |f(x)|)dx$