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What is on today

1 The Fundamental Theorem of Calculus

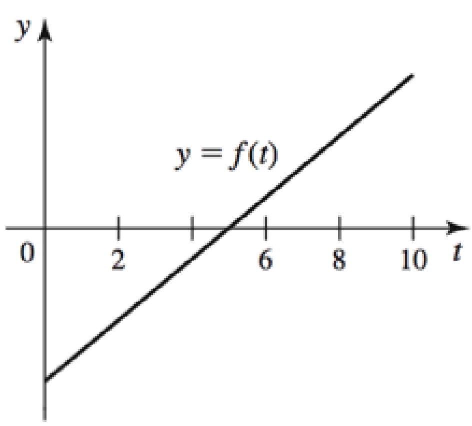
Briggs-Cochran-Gillett §5.3 pp. 362-376

Example 1 (§5.3 Ex. 70, 72)**.** *For each of the functions below, consider the function f given by its graph.*

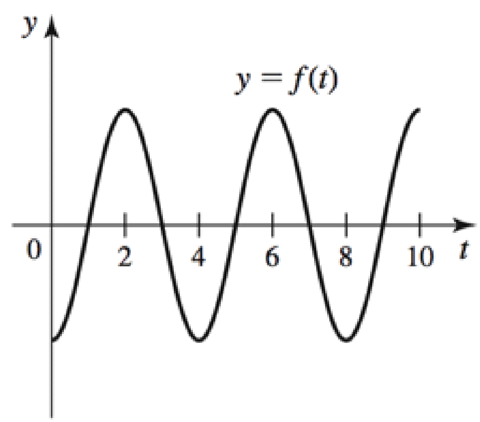
a. *Estimate the zeros of the net area function $N(x) = \int_0^x f(t)dt$ for $0 \le x \le 10$.*

b. *Estimate the points (if any) at which N has a local maximum or minimum.*

c. *Sketch a graph of N for $0 \le x \le 10$ (without a scale on the y-axis).*



1.



2.

1.1 The Fundamental Theorem of Calculus (FTC)

Let's look at the following example of different net area functions for a linear function:

Example 2 (§5.3 Ex. 17)**.** *Let $f(t) = t$ and consider the two net area functions $N(x) = \int_0^x f(t)dt$ and $F(x) = \int_2^x f(t)dt$.*

a. Evaluate $N(2)$ and $N(4)$. Then use geometry to find an expression for $N(x)$, $x \geq 0$.

b. Evaluate $F(4)$ and $F(6)$. Then use geometry to find an expression for $F(x)$, $x \geq 0$.

c. Show that $N(x) - F(x)$ is a constant and that $N'(x) = F'(x) = f(x)$.

This example suggests that the net area function $N(x)$ is an **antiderivative** of f. In fact, this holds for more general functions:

Theorem 3 (Fundamental Theorem of Calculus (FTC))**.** *Let f be a continuous function on $[a, b]$.*

1. *The net area function $N(x) = \int_a^x f(t)dt$ for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b) and we have*

$$N'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x),$$

which means the net area function of f is an antiderivative of f on $[a, b]$.

2. *If F is any antiderivative of f on $[a, b]$ then*

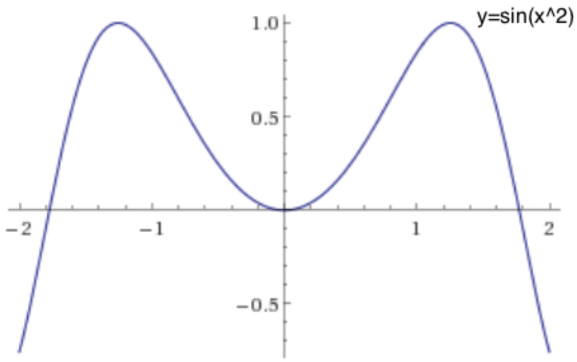
$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\big|_a^b.$$

Part 2 of the FTC is a powerful method for evaluating definite integrals and it is a direct consequence of part 1 (see p. 366 in the textbook).

Hence we have a new method to compute definite integrals $\int_a^b f(x)dx$:

- find any antiderivative of f; call it F;
- compute $F(b) - F(a)$.

Remark 4. *This method only works when we can find an antiderivative for f. This is only the case for a relatively small group of functions! The definition of integral is still the limit of Riemann sums and geometrically the net area between the graph and the x-axis in the given interval. You should always remember this! For example functions like $\sin(x^2)$ and e^{x^2} do not have an antiderivative but are continuous and hence integrable in any closed interval!*



Example 5 (§5.3 Ex. 64, 67, 68)**.** *Simplify the following expressions.*

a) $\dfrac{d}{dx}\displaystyle\int_{x^2}^{10} \frac{1}{z^2+1}dz$ *b)* $\dfrac{d}{dx}\displaystyle\int_{-x}^{x} \sqrt{1+t^2}dt$ *c)* $\dfrac{d}{dx}\displaystyle\int_{e^x}^{e^{2x}} \ln(t^2)dt$

Example 6 (§5.3 Ex. 30, 34, 40, 42, 50, 47)**.** *Evaluate the following integrals using the Fundamental Theorem of Calculus.*

1. $\displaystyle\int_0^2 (3x^2 + 2x)dx$

2. $\displaystyle\int_4^9 \frac{2 + \sqrt{t}}{t} dt$

3. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}}$

4. $\displaystyle\int_0^{\pi} (1 - \sin x)dx$

5. $\displaystyle\int_{\pi/16}^{\pi/8} 8\csc^2 4x dx$

6. $\displaystyle\int_0^{\pi/8} \cos 2x dx$