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What is on today

1 Substitution rule

Briggs-Cochran-Gillett §5.5 pp. 384-393

At the end of the last class, we integrated $\cos 2x$, which brings us to a useful strategy: substitution. If we were to guess at the value of the indefinite integral $\int \cos 2x dx$, perhaps we would start with $\int \cos x\, dx = \sin x + C$. We might incorrectly conclude that the indefinite integral of $\cos 2x$ is $\sin 2x + C$, but differentiation by the Chain Rule would reveal that

$$\frac{d}{dx}(\sin 2x + C) = 2\cos 2x \neq \cos 2x.$$

But now it's pretty clear that we were just off by a factor of 2 and that

$$\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C.$$

While this works here, this sort of trial-and-error approach is not practical for more complicated integrals, so we introduce the more systematic strategy of *substitution*, which we illustrate in the example of $\int \cos 2x dx$.

We first make the change of variable $u = 2x$. Then taking d's on both sides, this gives $du = 2dx$. We now rewrite our given integral:

$$\begin{aligned} \int \cos 2x dx &= \int \cos u \frac{du}{2} \\ &= \frac{1}{2}\int \cos u\, du \\ &= \frac{1}{2}\sin u + C \\ &= \frac{1}{2}\sin 2x + C. \end{aligned}$$

We formalize this process as follows:

Theorem 1 (Substitution rule for indefinite integrals)**.** *Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

In practice, we apply this theorem as follows:

Substitution Rule (Change of variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Rewrite the result in terms of x using $u = g(x)$.

Disclaimer: Not all integrals yield to the Substitution Rule!

Example 2 (§5.5 Ex. 18, 23, 30, 33, 36)**.** *Find the following indefinite integrals.*

1. $\displaystyle\int xe^{x^2}\,dx$

2. $\displaystyle\int x^3(x^4+16)^6\,dx$

3. $\displaystyle\int \frac{3}{1+25y^2}\,dy$

4. $\displaystyle\int \frac{x}{\sqrt{x-4}}\,dx$

5. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

1.1 Substitution Rule for definite integrals

We also have a Substitution Rule for computing definite integrals:

Theorem 3 (Substitution Rule for definite integrals)**.** *Let $u = g(x)$, where g' is continuous on $[a, b]$ and let f be continuous on the range of g. Then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 4 (§5.5 Ex. 40, 44, 54, 58, 66, 78)**.** *Compute the following integrals.*

1. $\displaystyle\int_0^2 \frac{2x}{(x^2+1)^2}\,dx$

2. $\displaystyle\int_0^4 \frac{p}{\sqrt{9+p^2}}\,dp$

3. $\displaystyle\int \sin^2 x\,dx$

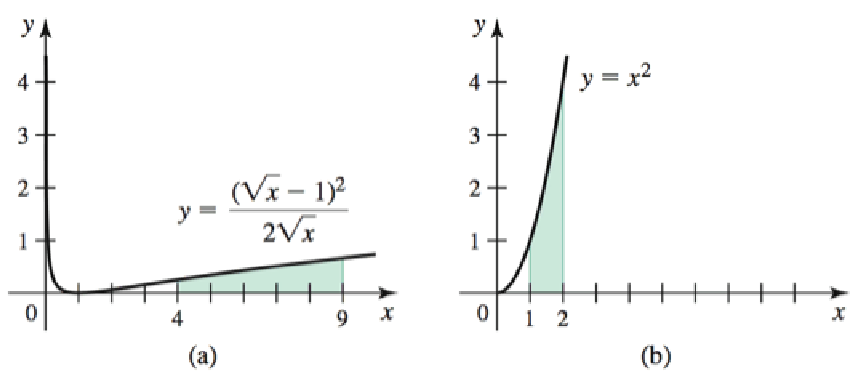
4. $\displaystyle\int x\cos^2(x^2)\,dx$

5. $\displaystyle\int (x^{3/2}+8)^5\sqrt{x}\,dx$

6. $\displaystyle\int_0^{\pi/4} e^{\sin^2 x}\sin(2x)\,dx$

Example 5 (§5.5 Ex. 82)**.** *Find the area of the region bounded by the graph of* $f(x) = \dfrac{x}{\sqrt{x^2-9}}$ *and the* x*-axis between* $x=4$ *and* $x=5$*.*

Example 6 (§5.5 Ex. 93)**.** *The area of the shaded region under the curve* $y = \dfrac{(\sqrt{x}-1)^2}{2\sqrt{x}}$ *on the interval* $[4,9]$ *in (a) equals the area of the shaded region under the curve* $y=x^2$ *on the interval* $[1,2]$ *in (b). Without computing areas, explain why.*



1.2 Further examples

Example 7 (§5.5 Ex. 94, 96). *Evaluate the following integrals:*

1. $\displaystyle\int \left(5f^3(x) + 7f^2(x) + f(x)\right)\ f'(x)\,dx$

2. $\displaystyle\int_0^1 f'(x)f''(x)\,dx$, *where* $f'(0) = 3$ *and* $f'(1) = 2$.

Example 8 (§5.5 Ex. 90 a). *True or False?*

1. $\displaystyle\int \tan(x)\,dx = -\ln|\cos(x)| + C$

2. $\displaystyle\int \tan(x)\,dx = \ln|\sec(x)| + C$

Here is a table of integrals you should know:

$\dfrac{d}{du}F(u) = f(u)$	$\displaystyle\int f(u)\,du = F(u) + C$
$\dfrac{d}{du}u^{n+1} = (n+1)u^n$	$\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
$\dfrac{d}{du}\ln u = \dfrac{1}{u}$	$\displaystyle\int \frac{1}{u}\,du = \ln\lvert u\rvert + C$
$\dfrac{d}{du}\sin u = \cos u$	$\displaystyle\int \cos u\,du = \sin u + C$
$\dfrac{d}{du}\cos u = -\sin u$	$\displaystyle\int \sin u\,du = -\cos u + C$
$\dfrac{d}{du}\tan u = \sec^2 u$	$\displaystyle\int \sec^2 u\,du = \tan u + C$
$\dfrac{d}{du}\sec u = \sec u \tan u$	$\displaystyle\int \sec u \tan u\,du = \sec u + C$
$\dfrac{d}{du}e^u = e^u$	$\displaystyle\int e^u\,du = e^u + C$
$\dfrac{d}{du}\sin^{-1}u = \dfrac{1}{\sqrt{1-u^2}}$	$\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C$
$\dfrac{d}{du}\tan^{-1}u = \dfrac{1}{1+u^2}$	$\displaystyle\int \frac{1}{1+u^2}\,du = \tan^{-1}u + C$
$\dfrac{d}{du}\sec^{-1}u = \dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}$	$\displaystyle\int \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\,du = \sec^{-1}u + C$