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What is on today

1 Linear independence

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Lay–Lay–McDonald §1.7 pp. 56 – 63

Today we take another look at the homogeneous equations $A\mathbf{x} = \mathbf{0}$ we considered in the previous class, this time writing the equations as vector equations. This perspective raises new questions about the vectors that appear in the vector equations.

As an example, consider the equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This equation has the trivial solution (where $x_1 = x_2 = x_3 = 0$), but does it have any others? This prompts us to make the following definition:

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is said to be *linearly independent* if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = x_2 = \dots = x_p = 0$. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be *linearly dependent* if there exist weights c_1, \dots, c_p not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0}.$$

Example 1. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

1. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

2. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Now suppose we begin with a matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$! Thus we have the following important fact:

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Example 2. Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

Example 3. Given $A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

Now we analyze linear independence of various sets of vectors:

- Suppose we have a set containing only one vector \mathbf{v} . This is linearly independent if and only if \mathbf{v} is not the zero vector. This is because the vector equation

$$x_1\mathbf{v} = \mathbf{0}$$

has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$. The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

- If we have a set containing two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$, then we can decide by inspection if the set is linearly dependent, as the following example illustrates.

Example 4. Determine if the following sets of vectors are linearly independent:

$$1. \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$2. \mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$3. \mathbf{v}_1 = \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

In geometric terms, two vectors are linearly dependent if and only if they lie on the same line through the origin.

Now we have a theorem to tell us about linear independence of larger sets of vectors:

Theorem 5 (Characterization of linearly dependent sets). *A set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some \mathbf{v}_j (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.*

Note that the above theorem does NOT say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Example 6. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by \mathbf{u} and \mathbf{v} and explain why a vector \mathbf{w} is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

Note that the previous example generalizes to any set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in \mathbb{R}^3 with \mathbf{u} and \mathbf{v} linearly independent. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ will be linearly dependent if and only if \mathbf{w} is in the plane spanned by \mathbf{u} and \mathbf{v} .

The next two theorems describe special cases in which the linear dependence of a set is automatic.

Theorem 7. *If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is linearly dependent if $p > n$.*

Proof. Let $A = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_p]$. Then A is $n \times p$:

$$A = \underbrace{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}}_p$$

and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of n equations in p unknowns. If $p > n$, there are more variables than equations, so there must be a free variable. Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and the columns of A are linearly dependent. \square

Example 8. Are the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ linearly independent?

Theorem 9. *If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n contains the zero vector, then the set is linearly dependent.*

Proof. By renumbering the vectors, we may suppose that $\mathbf{v}_1 = \mathbf{0}$. Then the equation

$$1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0},$$

shows that S is linearly dependent. □

Example 10. *Determine if the given set is linearly dependent:*

1. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

3. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

Example 11. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$.

1. Are the sets $\{\mathbf{u}, \mathbf{v}\}, \{\mathbf{u}, \mathbf{w}\}, \{\mathbf{u}, \mathbf{z}\}, \{\mathbf{v}, \mathbf{w}\}, \{\mathbf{v}, \mathbf{z}\}, \{\mathbf{w}, \mathbf{z}\}$ each linearly independent? Why or why not?
2. Does the answer to the previous question imply that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is linearly independent?
3. To determine if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is linearly dependent, is it wise to check if, say, \mathbf{w} is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{z}$?
4. Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ linearly dependent?