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What is on today

1 Linear independence

Lay–Lay–McDonald §1.7 pp. 56 – 63

We wrap up our discussion on linear independence by considering the following example:

Example 1. *Let* $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$.

1. *Are the sets* $\{\mathbf{u}, \mathbf{v}\}, \{\mathbf{u}, \mathbf{w}\}, \{\mathbf{u}, \mathbf{z}\}, \{\mathbf{v}, \mathbf{w}\}, \{\mathbf{v}, \mathbf{z}\}, \{\mathbf{w}, \mathbf{z}\}$ *each linearly independent? Why or why not?*

2. *Does the answer to the previous question imply that* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ *is linearly independent?*

3. *To determine if* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ *is linearly dependent, is it wise to check if, say,* $\mathbf{w}$ *is a linear combination of* $\mathbf{u}, \mathbf{v}, \mathbf{z}$*?*

4. *Is* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ *linearly dependent?*

2 Introduction to linear transformations

Lay–Lay–McDonald §1.8 pp. 63 – 69

Now we'll look at transforming vectors under matrix multiplication, which introduces the idea of *linear transformations*. For example, in the equation

$$\begin{array}{ccc} A & \mathbf{x} & = \mathbf{b} \\ \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \end{array}$$

multiplication by the matrix A transforms $\mathbf{x}$ into $\mathbf{b}$, and in the equation

$$\begin{array}{ccc} A & \mathbf{u} & = \mathbf{0} \\ \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} & = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

multiplication by A transforms $\mathbf{u}$ into $\mathbf{0}$.

From this new point of view, solving the equation $A\mathbf{x} = \mathbf{b}$ amounts to finding all vectors $\mathbf{x}$ in $\mathbb{R}^4$ that are transformed into the vector $\mathbf{b}$ in $\mathbb{R}^2$ under multiplication by A. Here we introduce some new terminology to further this viewpoint.

A *transformation* (or *function* or *mapping*) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector in $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the *domain* of T and $\mathbb{R}^m$ is called the *codomain* of T. The notation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the *image* of $\mathbf{x}$. The set of all images $T(\mathbf{x})$ is called the *range* of T.

Example 2. *Let* $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ *and define a transformation* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *by* $T(\mathbf{x}) = A\mathbf{x}$, *so that*

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}.$$

1. *Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.*

2. *Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.*

3. *Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?*

4. *Determine if $\mathbf{c}$ is in the range of the transformation T.*

The next two matrix transformations each have a nice geometric interpretation.

Example 3. *If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ projects points in $\mathbb{R}^3$ into the x_1x_2-plane, because*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Example 4. *Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A(\mathbf{x})$ is called a* shear transformation. *This transformation sends a square to a parallelogram, deforming the square as if the top of the square were pushed to the right while the base is held fixed.*

Recall that we saw earlier that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ has the properties

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}, \quad A(c\mathbf{u}) = cA\mathbf{u},$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars c. These key properties lead us to the formal definition of a linear transformation.

Definition 5. *A transformation T is* linear *if*

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ *for all* $\mathbf{u}, \mathbf{v}$ *in the domain of* T,
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ *for all scalars* c *and all* $\mathbf{u}$ *in the domain of* T.

Remark 6. *Note that every matrix transformation is a linear transformation.*

Here are a few more useful facts, both of which can be derived from the above. If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$ and $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d.

Example 7. *Given a scalar r, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$. T is called a contraction when $0 \leq r \leq 1$ and a dilation when $r > 1$. Let $r = 2$ and show that T is a linear transformation.*

Example 8. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. *Give a geometric description of the transformation* $\mathbf{x} \mapsto A\mathbf{x}$.

Example 9. *An affine transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(x) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and $\mathbf{b}$ in $\mathbb{R}^m$. Show that T is not a linear transformation when $\mathbf{b} \neq \mathbf{0}$. (Affine transformations are important in compute graphics.)*