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What is on today

1 The matrix of a linear transformation

Lay–Lay–McDonald §1.9 pp. 71 – 78

Whenever a linear transformation T arises geometrically, it's an interesting problem to compute the corresponding matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$. (Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is actually a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.) The key to finding A is to observe that T is completely determined by what it does to the columns of the $n \times n$ identity matrix I_n.

Example 1 *The columns of* $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *are* $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *Suppose T is a linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$ *such that* $T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ *and* $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. *Find a formula for the image of an arbitrary* $\mathbf{x}$ *in* $\mathbb{R}^2$.

Theorem 2 *Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear transformation. Then there exists a unique matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x} \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)]. \tag{1}$$

The matrix A in (1) is called the *standard matrix for the linear transformation T*.

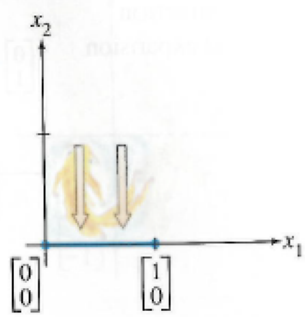
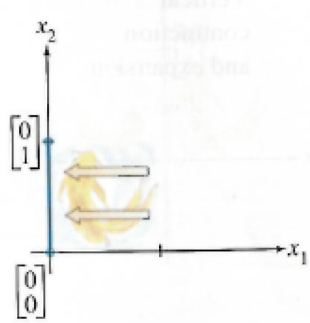
> Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as a matrix transformation, and vice versa!

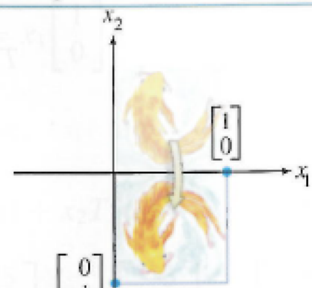
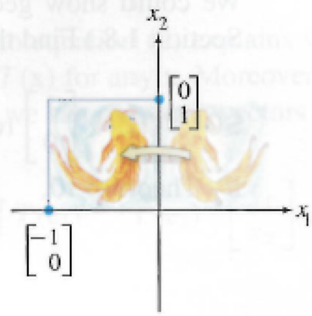
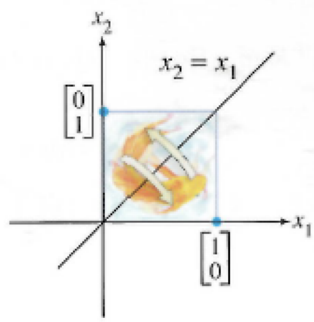
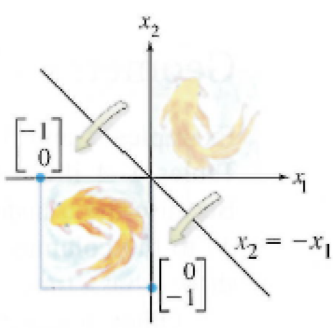
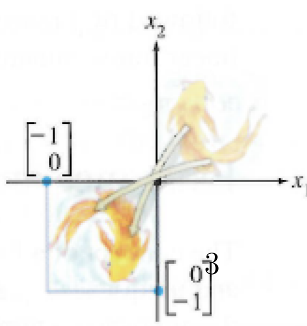
We practice with finding the standard matrix for linear transformations in the next two examples:

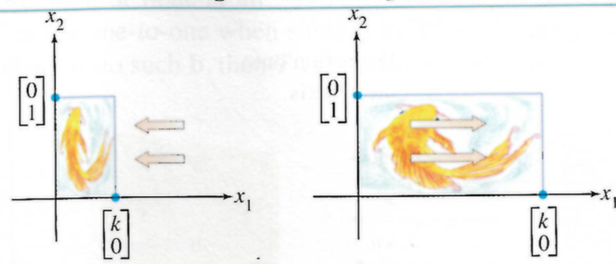
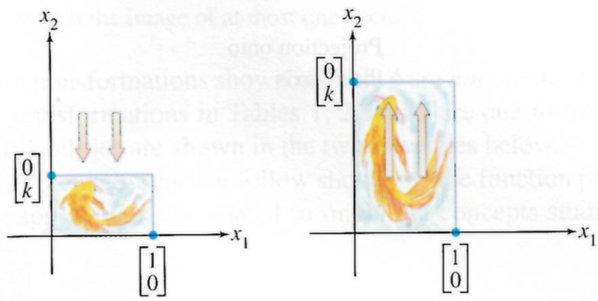
Example 3 *Find the standard matrix A for the dilation $T(\mathbf{x}) = 4\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^2$.*

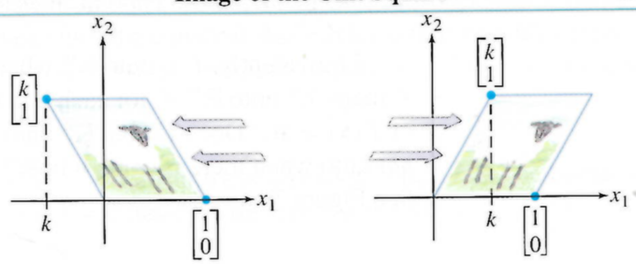
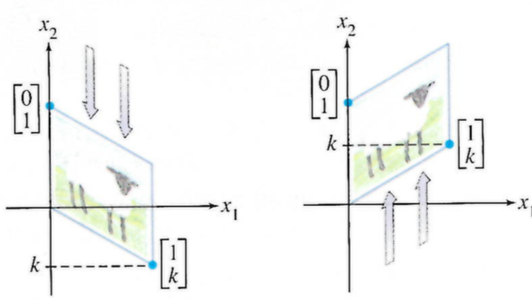
Example 4 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. Such a transformation is linear. Find the standard matrix A of this transformation.*

Below we reproduce some helpful figures from the textbook (§1.8, Tables 1–4) illustrating various geometric linear transformations (projections, reflections, contractions and cxpansions, and shears, respectively) of $\mathbb{R}^2$.

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$
Reflection through the origin		$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$ $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$ $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Now we translate our earlier existence and uniqueness questions about solutions of linear systems into questions about linear transformations, via the following terminology.

Definition 5 *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be* onto $\mathbb{R}^m$ *if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one $\mathbf{x} \in \mathbb{R}^n$.*

Equivalently, T is onto $\mathbb{R}^m$ when the range of T is all of the codomain $\mathbb{R}^m$. So the question "Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$?" is an existence question. The mapping T is not onto when there is some $\mathbf{b} \in \mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution.

Definition 6 *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be* one-to-one *if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one $\mathbf{x} \in \mathbb{R}^n$.*

Equivalently, T is one-to-one if, for each $\mathbf{b} \in \mathbb{R}^m$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or none at all. "Is T one-to-one?" is a uniqueness question.

Example 7 *Let T be the linear transformation whose standard matrix is $A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?*

Theorem 8 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.*

Theorem 9 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then*

1. *T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.*
2. *T is one-to-one if and only if the columns of A are linearly independent.*

Example 10 *Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?*