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What is on today

1 Matrix factorization

Lay–Lay–McDonald §2.5 pp. 125 – 129

Example 1. *Consider* $A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

Use this LU *factorization of* A *to solve* $A\mathbf{x} = \mathbf{b}$, *where* $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

The computational efficiency of the LU factorization depends on knowing L and U. The algorithm for finding the LU factorization shows that the row reduction of A to an echelon form U amounts to an LU factorization because it produces L with essentially no extra work.

Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another row *below* it. In this case, there exist unit lower triangular elementary matrices $E_1, \ldots E_p$ such that

$$E_p \cdots E_1 A = U. \tag{1}$$

Then

$$A = (E_p \cdots E_1)^{-1} U = LU \tag{2}$$

where $L = (E_p \cdots E_1)^{-1}$. It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.

Note that the row operations that reduce A to U in (1) also reduce the L to I in (2), because

$$E_p \cdots E_1 L = (E_p \cdots E_1)(E_p \cdots E_1)^{-1} = I.$$

This observation is the key to constructing L.

Algorithm for an LU factorization

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Step 1 is not always possible, but when it is, by the argument above, this implies that an LU factorization exists. In the next example, we will demonstrate how to implement Step 2.

Example 2. *Find an LU factorization of* $A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$.