Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Cramer's Rule, Volume, and Linear Transformations

Lay–Lay–McDonald §3.3 pp. 179 – 186

Today we give some formulas for using the determinant in various calculations, as well as a geometric interpretation of the determinant.

Cramer's Rule can be used to study how the solution of $A\mathbf{x} = \mathbf{b}$ changes as the entries of $\mathbf{b}$ change. To give the rule, we first define some notation:

For any $n \times n$ matrix A and any vector $\mathbf{b} \in \mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$:

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{b} \cdots \mathbf{a}_n],$$

where $\mathbf{b}$ takes the place of $\mathbf{a}_i$.

Theorem 1 (Cramer's rule)**.** *Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by*

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \ldots n.$$

Example 2. *Use Cramer's rule to solve the system*

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8. \end{aligned}$$

Cramer's rule also gives us a formula for the inverse of an $n \times n$ matrix A^{-1}. The jth column of A^{-1} is a vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{e}_j$, where $\mathbf{e}_j$ is the jth column of the identity matrix, and the ith entry of $\mathbf{x}$ is the (i, j)th entry of A^{-1}. By Cramer's rule, we have

$$(i, j)\text{th entry of } A^{-1} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}.$$

Recall that A_{ji} denotes the submatrix of A formed by deleting row j and column i. A cofactor expansion down column i of $A_i(\mathbf{e}_j)$ shows that

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji},$$

where C_{ji} is a cofactor of A. Thus we have the following formula:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_n n \end{bmatrix}. \tag{1}$$

The matrix of cofactors on the right side of (1) is called the *adjugate* of A, denoted by $\operatorname{adj} A$.

Theorem 3. *Let A be an invertible $n \times n$ matrix. Then*

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Example 4. *Find the inverse of the matrix* $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

Here is a result that tells us about a geometric interpretation of the determinant:

Theorem 5. *If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.*

Example 6. *Calculate the area of the parallelogram determined by the points* $(-2,-2),(0,3),(4,-1)$, *and* $(6,4)$.

Determinants can be used to describe an important geometric property of linear transformations in the plane and in $\mathbb{R}^3$. If T is a linear transformation and S is a set in the domain of T, let $T(S)$ denote the set of images of points in S. We are interested in how the area (or volume) of $T(S)$ compares with the area (or volume) of the original set S.

Theorem 7. *Let* $T:\mathbb{R}^2 \to \mathbb{R}^2$ *be the linear transformation determined by a* 2×2 *matrix* A. *If* S *is a parallelogram in* $\mathbb{R}^2$, *then*

$$\textit{area of } T(S) = |\det A| \cdot \textit{area of } S.$$

If T *is determined by a* 3×3 *matrix* A, *and if* S *is a parallelepiped in* $\mathbb{R}^3$, *then*

$$\textit{volume of } T(S) = |\det A| \cdot \textit{volume of } S.$$

It turns out that the conclusions of the above theorem hold whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

Example 8. *Let* a *and* b *be positive numbers. Find the area of the region* E *bounded by the ellipse whose equation is*

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$