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What is on today

1 Vector Spaces and Subspaces

Lay–Lay–McDonald §4.1 pp. 192 – 197

The work we've been doing with vectors in $\mathbb{R}^n$ can be understood in a more general framework once we have the notion of a *vector space*, which will be our object of study today.

Definition 1. *A (real)* vector space *is a nonempty set V of objects, called vectors, on which are defined two operations, addition and multiplication by scalars (real numbers), subject to the ten axioms below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and for all scalars c, d.*

1. $\mathbf{u} + \mathbf{v} \in V$.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. *There is a zero vector $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.*
5. *For each $\mathbf{u} \in V$, there is a vector $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.*
6. $c\mathbf{u} \in V$.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Note that the zero vector $\mathbf{0}$ is unique, and for each $\mathbf{u} \in V$, its negative $-\mathbf{u}$ is unique.

Example 2. *The spaces $\mathbb{R}^n$ for $n \geq 1$ are vector spaces.*

Example 3. *For $n \geq 0$, the set P_n of polynomials of degree at most n consists of all polynomials of the form*

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are real numbers. If $p(t) = a_0 \neq 0$, the degree of p is zero. If all of the coefficients are zero, p is called the zero polynomial. Show that P_n is a vector space.

Example 4. *Let V be the set of all real-valued functions defined on a set D (where D is $\mathbb{R}$ or some interval on the real line). Show that V is a vector space.*

In many problems, a vector space consists of a subset of vectors from some larger vector space. In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

Definition 5. *A* subspace *of a vector space V is a subset H of V that has three properties:*

1. *The zero vector of V is in H.*
2. *H is closed under vector addition: $\mathbf{u}, \mathbf{v} \in H \Rightarrow \mathbf{u} + \mathbf{v} \in H$.*
3. *H is closed under multiplication by scalars: if c is a scalar and $\mathbf{u} \in H$, then $c\mathbf{u} \in H$.*

Example 6. *Is the set consisting of the zero vector in a vector space V a subspace of V?*

Example 7. *Let P be the set of all polynomials with real coefficients, with the usual operations in P. Then P is a subspace of the space of all real-valued functions on $\mathbb{R}$. Also, for each $n \geq 0$, P_n is a subspace of P.*

Example 8. *The vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. However, the set $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \right\}$ where $s, t \in \mathbb{R}$ is a subset of $\mathbb{R}^3$. Show that H is a subspace of $\mathbb{R}^3$.*

Example 9. *Consider a plane in $\mathbb{R}^3$ not through the origin. Is it a subspace of $\mathbb{R}^3$?*

Example 10. *Let V be a vector space, and let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Let $H =$Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ be the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$. Show that H is a subspace of V.*

The argument in the previous example can be generalized to prove the following:

Theorem 11. *If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.*

We call $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ the subspace spanned (or generated) by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Given any subspace H of V, a spanning (or generating) set for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example 12. *Let H be the set of all vectors of the form $(a-3b, b-a, a, b)$, where a, b are arbitrary real numbers. Show that H is a subspace of $\mathbb{R}^4$.*

Example 13. *For what values of h will $\mathbf{y}$ be in the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ if* $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$*?*

Example 14. *Show that the set H of all points of $\mathbb{R}^2$ of the form $(3a, 2+5a)$ is not a vector space.*