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What is on today

1 The characteristic equation

Lay–Lay–McDonald §5.2 pp. 279 – 281

The next theorem presents one use of the characteristic polynomial and is helpful for iterative methods that approximate eigenvalues. We begin with some terminology. If A and B are $n \times n$ matrices, then we say that A is *similar to* B if there is an invertible matrix P such that

$$P^{-1}AP = B.$$

Writing $Q := P^{-1}$, we also have

$$Q^{-1}BQ = A.$$

So B is also similar to A, and we say that A *and* B *are similar*.

Theorem 1. *If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues with the same multiplicities.*

Proof. If $B = P^{-1}AP$ then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P.$$

We compute

$$\begin{aligned} \det(B - \lambda I) &= \det(P^{-1}(A - \lambda I)P) \\ &= \det(P^{-1}) \det(A - \lambda I) \det(P). \end{aligned}$$

Since $\det(P^{-1}) \det(P) = \det(P^{-1}P) = \det(I) = 1$, we see that $\det(B - \lambda I) = \det(A - \lambda I)$. □

Remark 2. *Note that matrices that have the same eigenvalues might not be similar: for instance, the matrices* $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ *and* $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ *have the same eigenvalues but are not similar.*

Remark 3. *Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.*

We can use eigenvalues and eigenvectors to analyze the evolution of a dynamical system.

Example 4. *Let* $A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$. *Analyze the long-term behavior of the dynamical system defined by* $\mathbf{x}_{k+1} = A\mathbf{x}_k$ *(*$k = 0, 1, 2, \ldots$*) with* $\mathbf{x}_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

2 Diagonalization

Lay–Lay–McDonald §5.3 pp. 283 – 288

Diagonal matrices make some computations much easier, as the following example illustrates:

Example 5. *Let* $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$. *What is* D^2*? What is* D^k*?*

If $A = PDP^{-1}$ for some invertible P and diagonal D, then A^k is also easy to compute.

Example 6. *Let* $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. *Find a formula for* A^k*, given that* $A = PDP^{-1}$*, where* $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

A square matrix A is said to be *diagonalizable* if A is similar to a diagonal matrix: that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D. The next result gives us a characterization of diagonalizable matrices and how to construct a factorization.

Theorem 7 (Diagonalization)**.** *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.*

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an *eigenvector basis* of $\mathbb{R}^n$.

Example 8. *Diagonalize the matrix* $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$*, if possible.*

Example 9. *Diagonalize the matrix* $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$*, if possible.*

The following theorem provides a sufficient condition for a matrix to be diagonalizable:

Theorem 10. *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

However, it is not necessary for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable!

Here is how we handle matrices whose eigenvalues are not distinct:

Theorem 11. *Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.*

1. *For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.*
2. *The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens iff a) the characteristic polynomial factors completely into linear factors and b) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.*
3. *If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.*

Example 12. *Diagonalize the matrix* $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$*, if possible.*