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1 Introduction

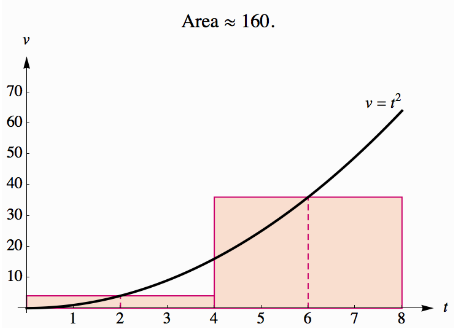
- Please refer to the syllabus for all course details.
- How lectures will work:
  - Notes will be available online (http://math.bu.edu/people/jbala/124.html) for use in each class.
  - We will use Learning Catalytics (through MyMathLab).
- This week:
  - Today we will review some topics in Chapter 5.
  - You will have a review worksheet covering topics in Chapter 5 during the first discussion session.
  - Let your TF know if you have not covered a specific topic.
- This course will focus on applications of integration, techniques for integration, sequences and series, and power series (Chapters 6 to 9 of the textbook).

2 Approximating area under curves and Riemann sums

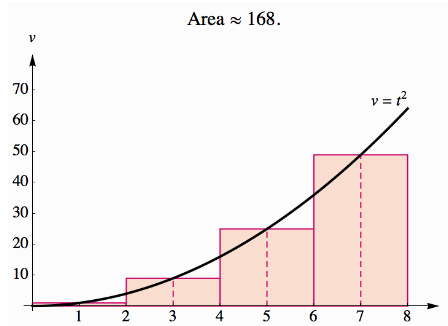
Briggs-Cochran-Gillett §5.1 pp. 333 - 347

Consider a car moving with velocity $v(t) = t^2\ mi/hr$, from $t = 0$ to $t = 8$. To calculate its displacement, we can approximate the area under the curve using rectangles:

Subdivision into 2 intervals:



Subdivision into 4 intervals:



To get better approximations, we could continue to subdivide the interval. Here we used the **midpoint** value for an approximate value of the velocity on the interval. We could also have used the **left endpoint** or the **right endpoint** (or any other point really...).

The process of **approximating the area below a curve** using areas of rectangles is known as computing **Riemann sums**. This is useful for much more than computing displacements.

Definition 1. *Given an interval* $[a, b]$ *and a positive integer* n*, we compute*

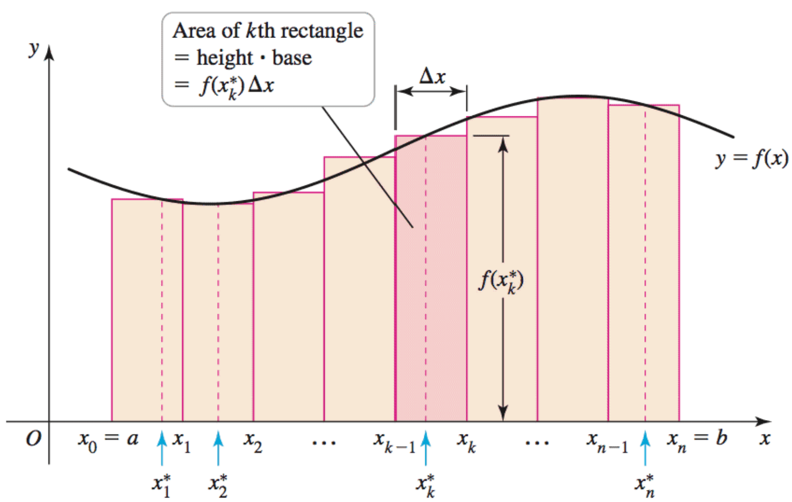
$$\Delta x = \frac{b-a}{n}.$$

Then we subdivide the interval into n *subintervals by letting*

$$x_0 = a \text{ and } x_k = x_{k-1} + \Delta x.$$

Then $x_n = b$*. The endpoints* $x_0, x_1, \ldots, x_n$ *of the subintervals are called* ***grid points****. The equally spaced numbers* x_k *form a* ***regular partition*** *of* $[a, b]$*.*

Let f be a function defined on an interval $[a, b]$ that we have partitioned into n intervals.



A **Riemann sum** is computed by adding the areas of any rectangles with bases in the subintervals in the partition and height equal to $f(x_k^*)$ where x_k^* is some point in the interval:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x.$$

- If x_k^* is the left endpoint of $[x_{k-1}, x_k]$ then we call it a **left Riemann sum**
- If x_k^* is the right endpoint of $[x_{k-1}, x_k]$ then we call it a **right Riemann sum**
- If x_k^* is the midpoint of $[x_{k-1}, x_k]$ then we call it a **midpoint Riemann sum**

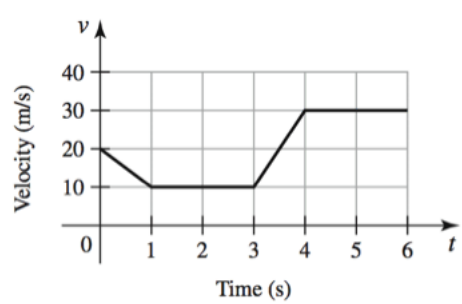
Example 2 (§5.1, Ex. 34)**.** *Let $f(x) = 4 - x$ on $[-1, 4]$, $n = 5$.*

1. *Sketch the graph of the function on the given interval.*
2. *Calculate Δx and the grid points $x_0, \ldots, x_n$.*
3. *Illustrate the midpoint Riemann sum by sketching the appropriate rectangles.*
4. *Calculate the midpoint Riemann sum.*

When we approximate areas under curves using Riemann sums, we can incrementally subdivide the interval into smaller and smaller pieces. This is a very important idea:

Theorem 3. *If f is a positive continuous function on $[a, b]$ then if we take smaller and smaller partitions of $[a, b]$, the Riemann sums are converging to a number that is the area under the curve between $x = a$ and $x = b$.*

Example 4 (§5.1, Ex. 66)**.** *Consider the velocity of an object moving along a line:*



1. *Use geometry to find the displacement of the object between $t = 0$ and $t = 3$.*
2. *Assuming that the velocity remains 30 m/s for $t \geq 4$, find the function that gives the displacement between $t = 0$ and any $t \geq 5$.*

Sigma notation can be used to express Riemann sums in a compact way. For example, the sum

$$1+2+3+\cdots+10$$

is written in sigma notation as

$$\sum_{k=1}^{10} k.$$

Here are two useful properties of sigma noation:

- Constant multiple rule: Let c be a constant. Then $\displaystyle\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$.
- Addition rule: $\displaystyle\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.

The following formulas for sums of powers of integers are also very useful:

Theorem 5 (Sums of powers of integers)**.** *Let n be a positive integer and c a real number.*

1. $\displaystyle\sum_{k=1}^{n} c = cn$
2. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Example 6 (§5.1 Ex. 41 g, h)**.** *Evaluate the following expressions:*

1. $\displaystyle\sum_{p=1}^{5}(2p+p^2)$

2. $\displaystyle\sum_{n=0}^{4}\sin\frac{n\pi}{2}$

3 Definite integrals

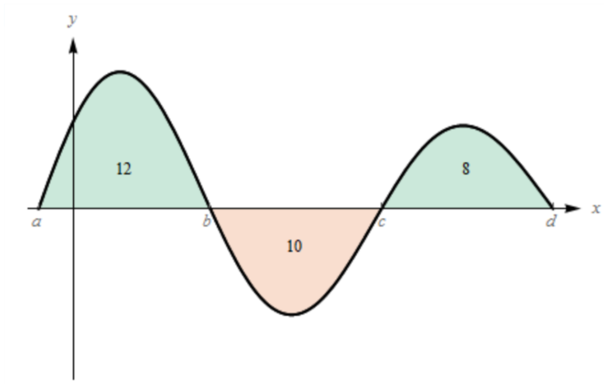
Briggs-Cochran-Gillett §5.2 pp. 348 - 361

Definition 7 (Definite integral)**.** *A function f defined on $[a,b]$ is integrable on $[a,b]$ if the limit $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists. This limit is the* ***definite integral of*** *f* ***from*** *a* ***to*** *b, which we write*

$$\int_a^b f(x)dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Definition 8 (Net area)**.** *Let R be the region bounded by a continuous function f and the x-axis between $x = a$ and $x = b$. The net area of R is the sum of the areas of the parts of R that lie above the x-axis minus the area of the parts of R that lie below the x-axis on $[a,b]$.*

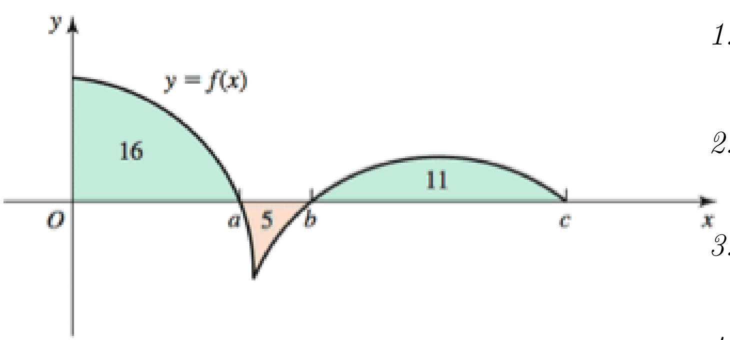
Geometrically: The *definite integral* corresponds to the net area:



We have

$$\int_a^b f(x)dx = 12$$
$$\int_b^c f(x)dx = -10$$
$$\int_a^d f(x)dx = 10.$$

Example 9 (§5.2 Ex. 33, 34, 35, 36)**.** *The figure shows the areas of regions bounded by the graph of f and the x-axis. Evaluate the following integrals.*



1. $\int_0^a f(x)dx$
2. $\int_0^b f(x)dx$
3. $\int_a^c f(x)dx$
4. $\int_0^c f(x)dx$

Here are some very important properties of definite integrals:

Let f and g be integrable functions on an interval that contains a, b, and p.

1. $\int_a^a f(x)\,dx = 0$ Definition
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition
3. $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$ For any constant c
5. $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

4 The Fundamental Theorem of Calculus

Briggs-Cochran-Gillett §5.3 pp. 362-376

Theorem 10 (Fundamental Theorem of Calculus)**.** *Let f be a continuous function on $[a, b]$.*

1. *The net area function $N(x) = \int_a^x f(t)dt$ for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b) and we have*

$$N'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x),$$

which means the net area function of f is an antiderivative of f on $[a, b]$.

2. *If F is any antiderivative of f on $[a, b]$ then*

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b.$$

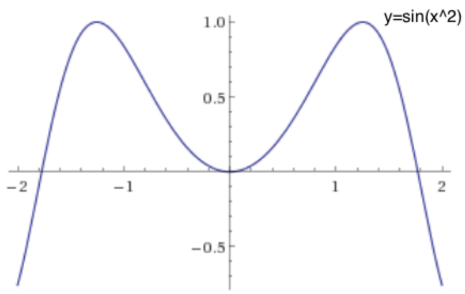
Part 2 of the FTC is a powerful method for evaluating definite integrals and it is a direct consequence of part 1 (see p. 366 in the textbook).

Hence we have a new method to compute definite integrals $\int_a^b f(x)dx$:

- find any antiderivative of f; call it F;

- compute $F(b) - F(a)$.

Remark 11. *This method only works when we can find an antiderivative for f. This is only the case for a relatively small group of functions! The definition of integral is still the limit of Riemann sums and geometrically the net area between the graph and the x-axis in the given interval. You should always remember this! For example functions like $\sin(x^2)$ and e^{x^2} do not have an antiderivative but are continuous and hence integrable in any closed interval!*



Example 12 (§5.3 Ex. 30, 34, 40, 47)**.** *Evaluate the following integrals using the Fundamental Theorem of Calculus.*

1. $\displaystyle\int_0^2 (3x^2 + 2x)dx$

2. $\displaystyle\int_4^9 \frac{2+\sqrt{t}}{t}dt$

3. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

4. $\displaystyle\int_0^{\pi/8} \cos 2x dx$

5 Substitution rule

Briggs-Cochran-Gillett §5.5 pp. 384-393

In the last set of examples, we integrated $\cos 2x$, which brings us to a useful strategy: substitution. If we were to guess at the value of the indefinite integral $\int \cos 2x dx$, perhaps we would start with $\int \cos x\, dx = \sin x + C$. We might incorrectly conclude that the indefinite integral of $\cos 2x$ is $\sin 2x + C$, but differentiation by the Chain Rule would reveal that

$$\frac{d}{dx}(\sin 2x + C) = 2\cos 2x \neq \cos 2x.$$

But now it's pretty clear that we were just off by a factor of 2 and that

$$\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C.$$

While this works here, this sort of trial-and-error approach is not practical for more complicated integrals, so we introduce the more systematic strategy of *substitution*, which we illustrate in the example of $\int \cos 2x dx$.

We first make the change of variable $u = 2x$. Then taking d's on both sides, this gives $du = 2dx$. We now rewrite our given integral:

$$\begin{aligned} \int \cos 2x dx &= \int \cos u \frac{du}{2} \\ &= \frac{1}{2}\int \cos u\, du \\ &= \frac{1}{2}\sin u + C \\ &= \frac{1}{2}\sin 2x + C. \end{aligned}$$

We formalize this process as follows:

Theorem 13 (Substitution rule for indefinite integrals)**.** *Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

In practice, we apply this theorem as follows:

Substitution Rule (Change of variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.

2. Substitute $u = g(x)$ and $du = g'(x)dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Rewrite the result in terms of x using $u = g(x)$.

Disclaimer: Not all integrals yield to the Substitution Rule!

Example 14 (§5.5 Ex. 18, 23, 30)**.** *Find the following indefinite integrals.*

1. $\int xe^{x^2}\,dx$

2. $\int x^3(x^4+16)^6\,dx$

3. $\int \frac{3}{1+25y^2}\,dy$

We also have a Substitution Rule for computing definite integrals:

Theorem 15 (Substitution Rule for definite integrals)**.** *Let $u = g(x)$, where g' is continuous on $[a, b]$ and let f be continuous on the range of g. Then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 16 (§5.5 Ex. 40, 44)**.** *Compute the following integrals.*

1. $\int_0^2 \frac{2x}{(x^2+1)^2}\,dx$

2. $\displaystyle\int_0^4 \frac{p}{\sqrt{9+p^2}}\,dp$

Example 17 (§5.5 Ex. 82)**.** *Find the area of the region bounded by the graph of* $f(x) = \dfrac{x}{\sqrt{x^2-9}}$ *and the* x*-axis between* $x = 4$ *and* $x = 5$*.*

Here is a table of integrals you should know:

$\frac{d}{du}F(u) = f(u)$	$\int f(u)\,du = F(u) + C$
$\frac{d}{du}u^{n+1} = (n+1)u^n$	$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
$\frac{d}{du}\ln u = \frac{1}{u}$	$\int \frac{1}{u}\,du = \ln\lvert u\rvert + C$
$\frac{d}{du}\sin u = \cos u$	$\int \cos u\,du = \sin u + C$
$\frac{d}{du}\cos u = -\sin u$	$\int \sin u\,du = -\cos u + C$
$\frac{d}{du}\tan u = \sec^2 u$	$\int \sec^2 u\,du = \tan u + C$
$\frac{d}{du}\sec u = \sec u \tan u$	$\int \sec u \tan u\,du = \sec u + C$
$\frac{d}{du}e^u = e^u$	$\int e^u\,du = e^u + C$
$\frac{d}{du}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}$	$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C$
$\frac{d}{du}\tan^{-1}u = \frac{1}{1+u^2}$	$\int \frac{1}{1+u^2}\,du = \tan^{-1}u + C$
$\frac{d}{du}\sec^{-1}u = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}$	$\int \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\,du = \sec^{-1}u + C$