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## What is on today

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## 1 Velocity and net change

Briggs-Cochran-Gillett §6.1 pp. 398 - 406

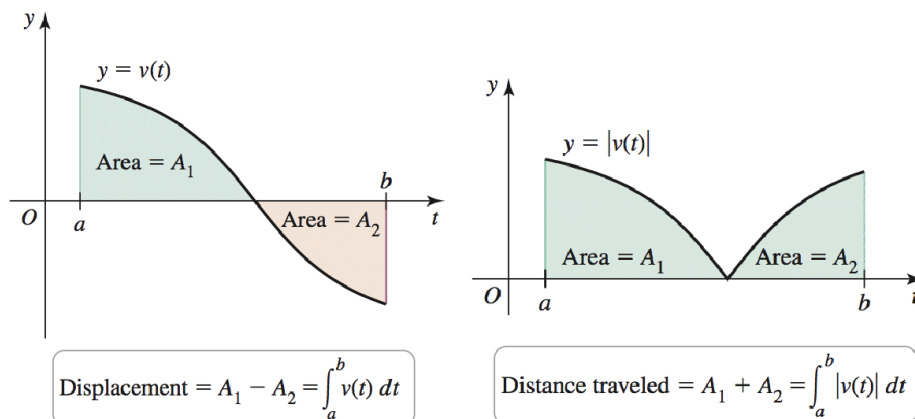
Suppose you are driving along a straight highway and your position relative to a reference point (or origin) is  $s(t)$  for  $t \geq 0$ . Your displacement over a time interval  $[a, b]$  is the change in position  $s(b) - s(a)$ . Now assume that  $v(t)$  is the velocity of the object at  $t$ . Recall that  $v(t) = s'(t)$ , so by the FTC, we have

$$\int_a^b v(t) dt = \int_a^b s'(t) dt = s(b) - s(a) = \mathbf{displacement}.$$

We see that the definite integral  $\int_a^b v(t) dt$  is the displacement (change in position) between times  $t = a$  and  $t = b$ . Equivalently, the displacement over the time interval  $[a, b]$  is the net area under the velocity curve over  $[a, b]$ .

Not to be confused with displacement is the distance traveled over a time interval, which is the total distance traveled by the object, independent of the direction of motion. If the velocity is positive, the object moves in the positive direction, and the displacement equals the distance traveled. However, if the velocity changes sign, then the displacement and the distance traveled are not generally equal. To compute the distance traveled, we need the magnitude of the velocity  $|v(t)|$ , which is called **speed**. We have that

$$\mathbf{distance\ traveled} = \int_a^b |v(t)| dt.$$



**Example 1** (§6.1 Ex. 12). Let  $v(t) = -t^2 + 5t - 4$  on  $0 \leq t \leq 5$ . Assume  $t$  is time measured in seconds and velocities have units of m/s.

1. Graph the velocity function over the given interval. Then determine when the motion is in the positive direction and when it is in the negative direction.
2. Find the displacement over the given interval.
3. Find the distance traveled over the given interval.

To find the displacement of an object, we do not need to know its initial position. What happens if we are interested in the actual position of the object at a future time? Suppose we know the velocity of an object and its initial position  $s(0)$ . The FTC gives us the answer:

$$\int_0^t v(x)dx = \int_0^t s'(x)dx = s(x)\Big|_0^t = s(t) - s(0).$$

Rearranging this gives that

$$s(t) = s(0) + \int_0^t v(x)dx.$$

**Example 2** (§6.1 Ex. 24). The velocity (in mi/hr) of a hiker walking along a straight trail is given by  $v(t) = 3 \sin^2(\pi t/2)$ , for  $0 \leq t \leq 4$ . Assume that  $s(0) = 0$  and  $t$  is measured in hours.

1. Determine the position function, for  $0 \leq t \leq 4$ . (Hint:  $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$ .)
2. What is the distance traveled by the hiker in the first 15 min of the hike?
3. What is the hiker's position at  $t = 3$ ?

Just like we calculated position from velocity, we can calculate velocity from acceleration. Given the acceleration  $a(x)$ , we have

$$\int_0^t a(x)dx = \int_0^t v'(x)dx = v(t) - v(0),$$

which gives

$$v(t) = v(0) + \int_0^t a(x)dx.$$

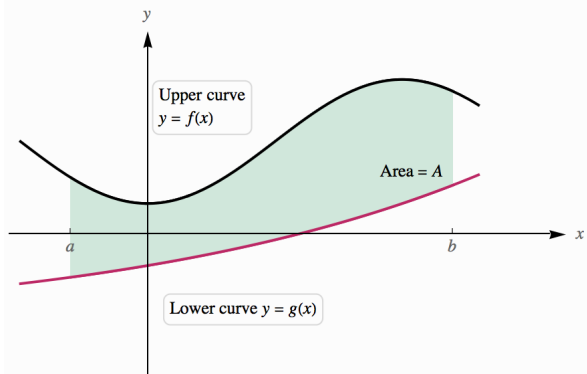
**Example 3** (§6.1 Ex. 36). A car slows down with an acceleration of  $a(t) = -15ft/s^2$ . Assume that  $v(0) = 60ft/s$ ,  $s(0) = 0$  and  $t$  is measured in seconds.

1. Determine and graph the position function, for  $t \geq 0$ .
2. How far does the car travel in the time it takes to come to rest?

## 2 Regions between curves

Briggs-Cochran-Gillett §6.2 pp. 412 - 416

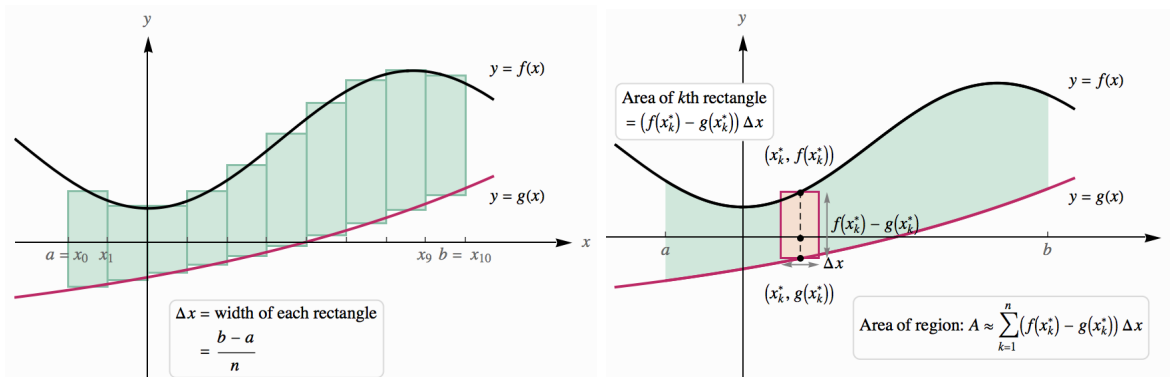
Here we generalize the method for finding the area of a region bounded by a single curve to regions bounded by two or more curves. Consider two functions  $f$  and  $g$  continuous on an interval  $[a, b]$  on which  $f(x) \geq g(x)$ . The goal is to find the area of the region bounded by the two curves and the lines  $x = a$  and  $x = b$ .



We compute the area using Riemann sums: partition the interval  $[a, b]$  into  $n$  subintervals using uniformly spaced grid points separated by a distance  $\Delta x = (b - a)/n$ . On each subinterval, we draw a rectangle extending from the lower curve to the upper curve. On the  $k$ th subinterval, we choose a point  $x_k^*$ , and the height of the corresponding rectangle is  $f(x_k^*) - g(x_k^*)$ . Therefore, the area of the  $k$ th rectangle is  $(f(x_k^*) - g(x_k^*))\Delta x$ . Summing

the areas of the  $n$  rectangles gives an approximation to the area of the region between the curves:

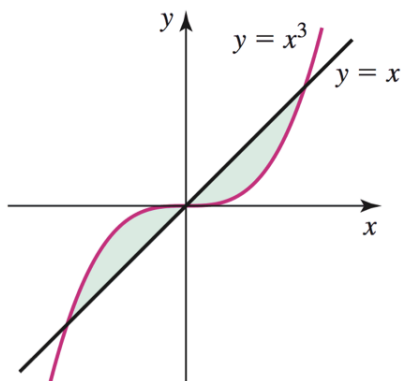
$$A \approx \sum_{k=1}^n (f(x_k^*) - g(x_k^*)) \Delta x.$$



As the number of grid points increases,  $\Delta x$  approaches zero, and these sums approach the area of the region between the curves:

**Definition 4.** Suppose that  $f$  and  $g$  are continuous functions with  $f(x) \geq g(x)$  on the interval  $[a, b]$ . The area of the region bounded by the graphs of  $f$  and  $g$  on  $[a, b]$  is  $A = \int_a^b (f(x) - g(x)) dx$ .

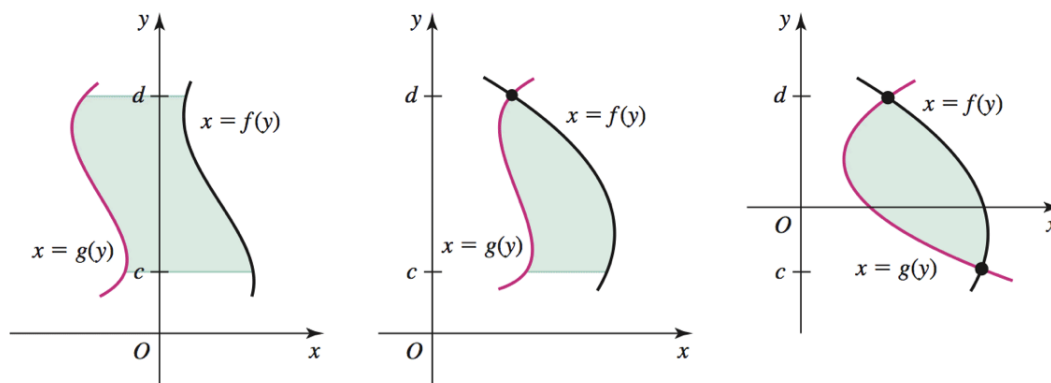
**Example 5** (§6.2 Ex. 6). Determine the area of the shaded region.



**Example 6** (§6.2 Ex. 10). Consider the region bounded by  $y = \cos x$  and  $y = \sin x$  between  $x = \pi/4$  and  $x = 5\pi/4$ . Sketch the region and find its area.

**Example 7** (§6.2 Ex. 19). Consider the region bounded by  $y = 2 - |x|$  and  $y = x^2$ . Sketch the region and find its area.

There are occasions when it is convenient to reverse the roles of  $x$  and  $y$ . Consider the regions shown below that are bounded by the graphs of  $x = f(y)$  and  $x = g(y)$ , where  $f(y) \geq g(y)$ , for  $c \leq y \leq d$  (which implies that the graph of  $f$  lies to the right of the graph of  $g$ ). The lower and upper boundaries of the regions are  $y = c$  and  $y = d$ , respectively.

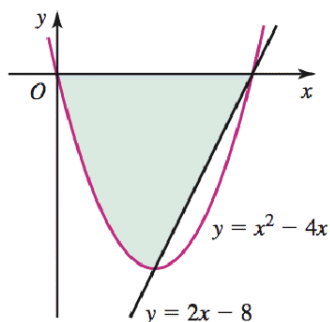


In cases like these, we treat  $y$  as the independent variable and divide the interval  $[c, d]$  into  $n$  subintervals of width  $\Delta y = (d - c)/n$ . We compute the Riemann sum and take the limit to obtain the following:

**Definition 8.** Suppose that  $f$  and  $g$  are continuous functions with  $f(y) \geq g(y)$  on the interval  $[c, d]$ . The area of the region bounded by  $x = f(y)$  and  $x = g(y)$  on  $[c, d]$  is

$$A = \int_c^d (f(y) - g(y)) dy.$$

**Example 9** (§6.2 Ex. 28). Express the area of the shaded region in terms of (a) one or more integrals with respect to  $x$  and (b) one or more integrals with respect to  $y$ .



**Example 10** (§6.2 Ex. 38). Consider the region bounded by  $x = y^2 - 4$  and  $y = x/3$ . Use any method (including geometry) to find the area of the region. Sketch the bounding curves and the region in question.

**Example 11** (§6.2 Ex. 52). Find the area of the region.

