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What is on today

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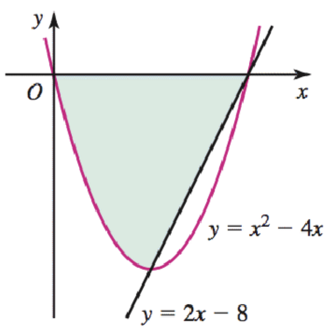
1 Regions between curves, wrap up

Briggs-Cochran-Gillett §6.2 pp. 412 - 416

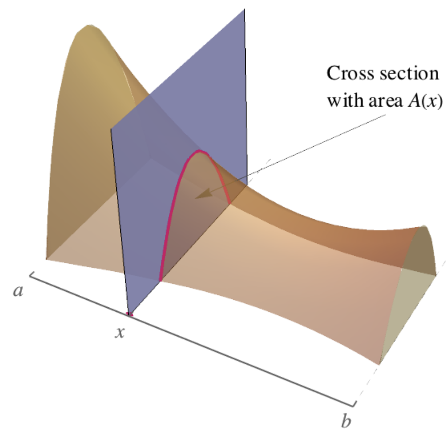
Example 1 (§6.2 Ex. 28). *Express the area of the shaded region in terms of (a) one or more integrals with respect to x and (b) and one or more integrals with respect to y.*



2 Volume by slicing

Briggs-Cochran-Gillett §6.3 pp. 420 - 429

Consider a solid object that extends in the x-direction from $x = a$ to $x = b$. Imagine cutting through the solid, perpendicular to the x-axis at a particular point x, and suppose the area of the cross section is given by a function A.



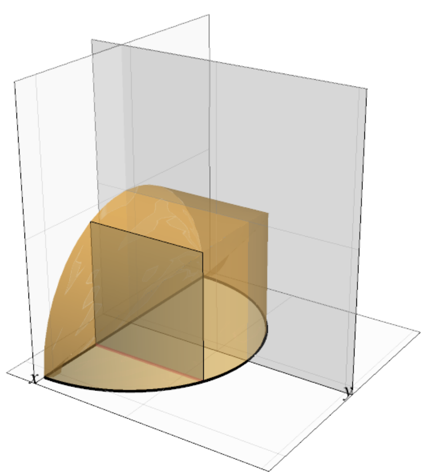
We give a formula to calculate the volume of this object, which is the basis of other volume formulas in this section:

General slicing method

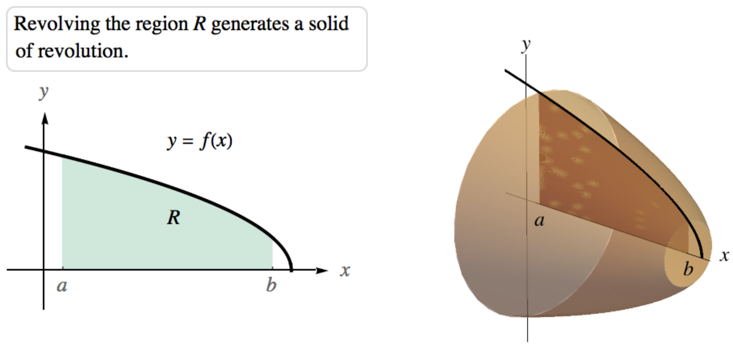
Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. The volume of the solid is

$$V = \int_a^b A(x)dx.$$

Example 2 (§6.3 Ex. 8). *Use the slicing method to calculate the volume of the solid whose base is the region bounded by the semicircle $y = \sqrt{1-x^2}$ and the x-axis, and whose cross sections through the solid perpendicular to the x-axis are squares.*



We can also consider a specific type of solid known as a **solid of revolution**. Suppose f is a continuous function with $f(x) \geq 0$ on $[a, b]$. Let R be the region bounded by the graph of f, the x-axis, the the lines $x = a$ and $x = b$. Now rotate R around the x-axis. As R revolves once about the x-axis, it sweeps out a 3-dimensional solid of revolution.



We find the volume of this solid using the general slicing method. Here the cross-sectional area function has a special form since all cross-sections perpendicular to the x-axis are circular disks with radius $f(x)$. Thus the area of a cross-section is

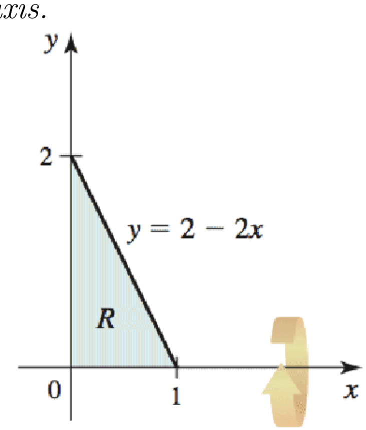
$$A(x) = \pi(f(x))^2.$$

Here we summarize what just happened, which we call the disk method:

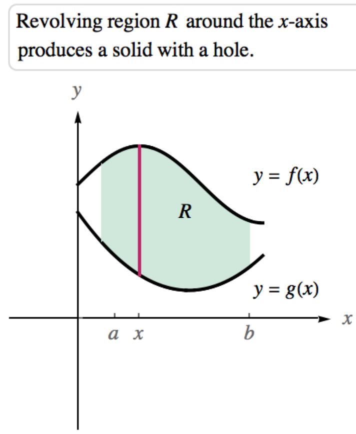
Disk method about the x-axis
Let f be continuous with $f(x) \geq 0$ on $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

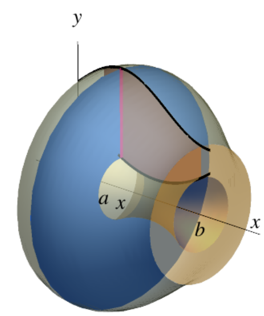
$$V = \int_a^b \pi(f(x))^2 dx.$$

Example 3 (§6.3 Ex. 18)**.** *Let R be the region bounded by the curves $y = 2-2x, y = 0, x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about the x-axis.*



A slight variation on the disk method enables us to compute the volume of more exotic solids of revolution. Suppose R is the region bounded by the graphs of f and g between $x = a$ and $x = b$, where $f(x) \geq g(x) \geq 0$. If R is revolved about the x-axis to create a solid of revolution, the solid generally has a hole through it.





Here a cross-section through the solid perpendicular to the x-axis is a circular washer with an outer radius of $R = f(x)$ and an inner radius of $r = g(x)$, where $a \leq x \leq b$. The area of the cross section is the area of the entire disk minus the area of the hole, or

$$A(x) = \pi(R^2 - r^2) = \pi((f(x))^2 - (g(x))^2).$$

This gives the washer method:

Washer method about the x-axis
Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x), y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi((f(x))^2 - (g(x))^2)dx.$$

Note that when $g(x) = 0$ in the washer method, the result is the disk method!

One can also revolve solids about the y-axis, and the resulting formulas for volume are as follows:

Disk and washer methods about the y-axis
Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y), x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi((p(y))^2 - (q(y))^2)dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi(p(y))^2 dy.$$

Example 4 (§6.3 Ex. 38)**.** *Consider the region bounded by* $y = \sqrt{x}, y = 0, x = 4$ *revolved about the* y*-axis. Find the volume of the solid of revolution.*

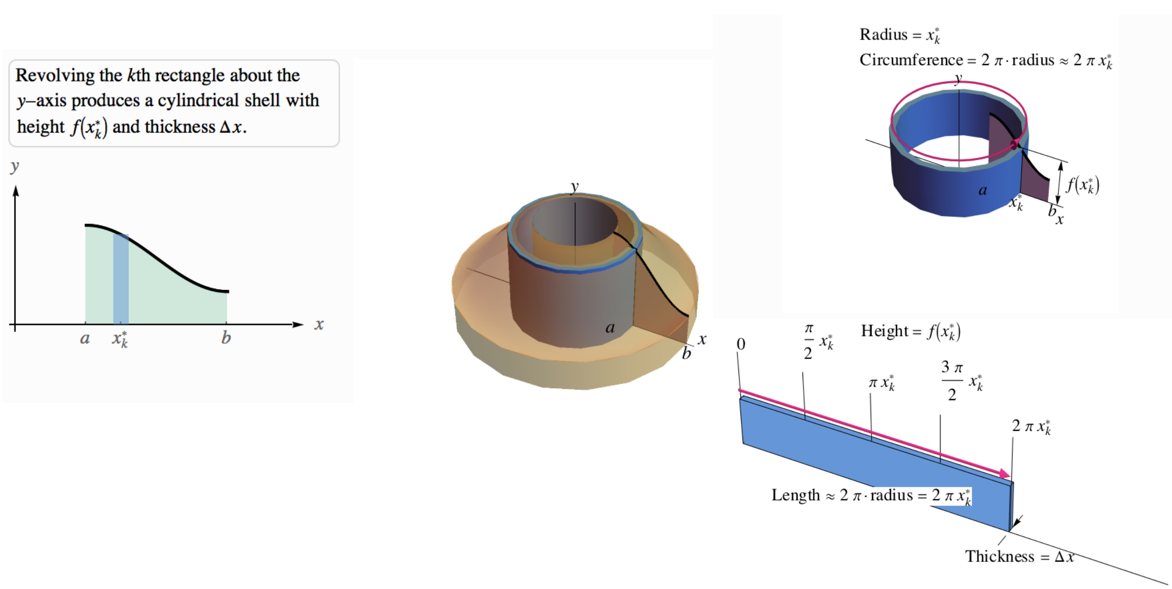
Example 5 (§6.3 Ex. 54)**.** *Consider the region bounded by* $y = (\ln x)/\sqrt{x}, y = 0,$ *and* $x = 2$ *revolved about the* x*-axis. Find the volume of the solid of revolution.*

3 Volume by shells

Briggs-Cochran-Gillett §6.4 pp. 434 - 441

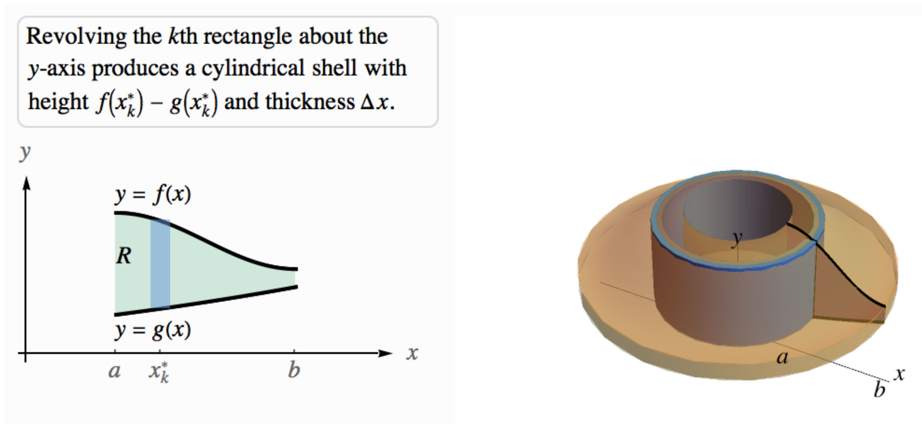
One can solve many challenging problems using the disk/washer method. There are, however, some volume problems that are difficult to solve with this method. For this reason, we also consider the shell method, which computes volumes via cylindrical shells.

Let R be a region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$. When R is revolved about the y-axis, a solid of revolution is formed. The idea is to compute the volume by computing the sum of volumes of thin cylindrical shells. In the figure below, you can imagine each cylindrical shell being unwrapped, giving a thin rectangular slab.



The approximate length of the slab is the circumference of a circle with radius x_k^*, which is $2\pi x_k^*$. The height of the slab is the height of the original rectangle $f(x_k^*)$ and its thickness is Δx. Therefore, the volume of the kth shell is approximately $2\pi x_k^* f(x_k^*)\Delta x$.

If the region R is bounded by two curves, $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ on $[a, b]$, we have that the volume of the kth shell is approximately $2\pi x_k^*(f(x_k^*) - g(x_k^*))\Delta x$.

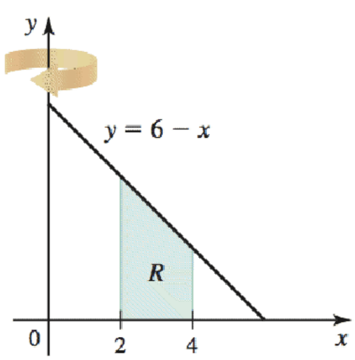


Summing over all of the shells and taking the limit as $\Delta x \to 0$ gives the following formula:

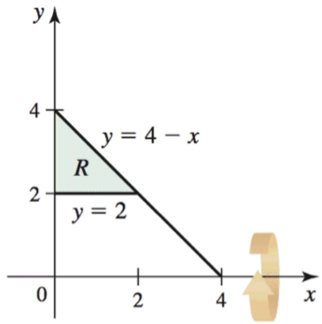
Volume by the shell method
Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b 2\pi x(f(x) - g(x))dx.$$

Example 6 (§6.4 Ex. 8)**.** *Let R be the region bounded by the curves $y = 6 - x, y = 0, x = 2$, and $x = 4$. Use the shell method to find the volume of the solid generated when R is revolved about the y-axis.*



Example 7 (§6.4 Ex. 17)**.** *Let R be the region bounded by the curves $y = 4 - x, y = 2$, and $x = 0$. Use the shell method to find the volume of the solid generated when R is revolved about the x-axis.*



Example 8 (§6.4 Ex. 58)**.** *Find the volume of the solid formed when the region bounded by $y = x^3$, the x-axis, and $x = 2$ is revolved about the x-axis using the method of your choice.*