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What is on today

1 Partial fractions

Briggs-Cochran-Gillett §7.5 pp. 523 - 529

In this section, we use partial fraction decomposition to integrate certain rational functions. The idea behind it is that it is straightforward to integrate

$$\int \frac{1}{x-2} + \frac{2}{x+4}\, dx,$$

but it's not quite obvious how to integrate

$$\int \frac{3x}{x^2+2x-8}\, dx.$$

However, it turns out that

$$\frac{3x}{x^2+2x-8} = \frac{1}{x-2} + \frac{2}{x+4},$$

so if we are able to compute the decomposition on the RHS, we can integrate.

To get the decomposition, the first step is to factor the denominator on the LHS:

$$x^2+2x-8 = (x-2)(x+4).$$

So our goal is to write

$$\frac{3x}{x^2+2x-8} = \frac{A}{x-2} + \frac{B}{x+4},$$

and we need to solve for A and B.

Multiplying both sides of the above by x^2+2x-8, we get

$$3x = A(x+4) + B(x-2),$$

and we compare the linear terms (terms with x) and constant terms (terms without x): we have

$$3x = Ax + Bx \;\Rightarrow 3 = A + B$$

and

$$0 = 4A - 2B.$$

Solving this system gives $A = 1$ and $B = 2$.

Example 1 (§7.5 Ex 14)**.** *Evaluate* $\int \frac{8}{(x-2)(x+6)}\,dx$.

Example 2 (§7.5 Ex 20)**.** *Evaluate* $\int \frac{y+1}{y^3+3y^2-18y}\,dy$.

Example 3 (§7.5 Ex 54)**.** *Find the area of the region bounded by the curves* $y = 1/x, y = x/(3x+4)$, *and the line* $x = 10$.

2 Numerical integration

Briggs-Cochran-Gillett §7.7 pp. 557 - 566

Sometimes we aren't able to analytically compute definite integrals, and in these situations, we rely on numerical methods. These methods provide approximations that are generally quite accurate. Because numerical methods do not typically produce exact results, we should understand the accuracy of approximations and bound the error.

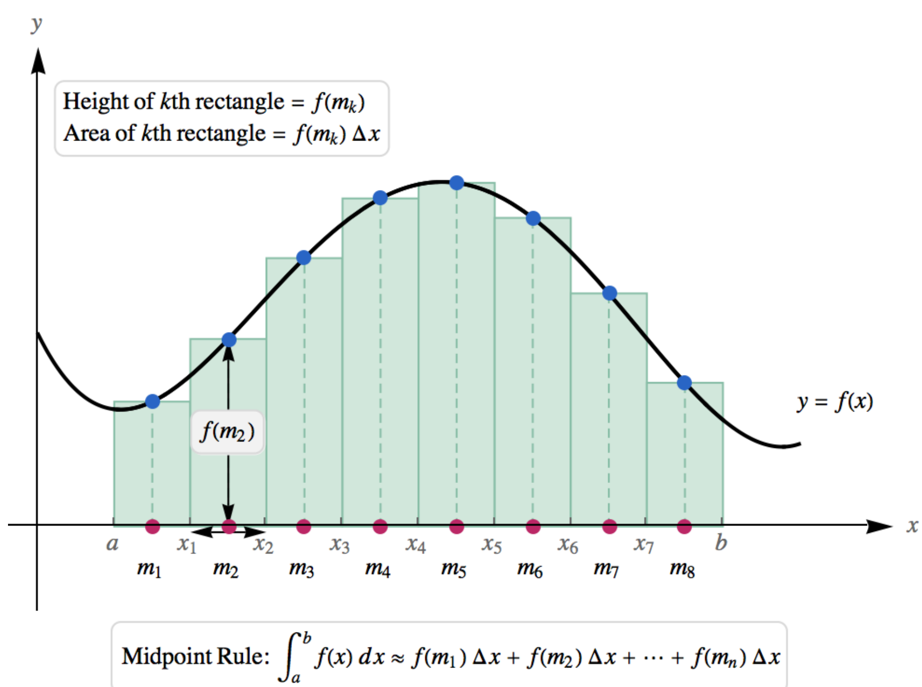
Definition 4. *Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common measures of the error in c as an approximation to x:*

- *absolute error:* $|c - x|$
- *relative error:* $\frac{|c-x|}{|x|}$

Example 5 (§7.7 Ex 8)**.** *Let $x = \sqrt{2}$ and $c = 1.414$. Compute the absolute and relative errors in using c to approximate x.*

Many numerical integration methods are based on the ideas that underlie Riemann sums; these methods approximate the net area of regions bounded by curves.

The Midpoint Rule approximates the region under the curve using rectangles, where the height of the kth rectangle uses the midpoint of the kth subinterval.



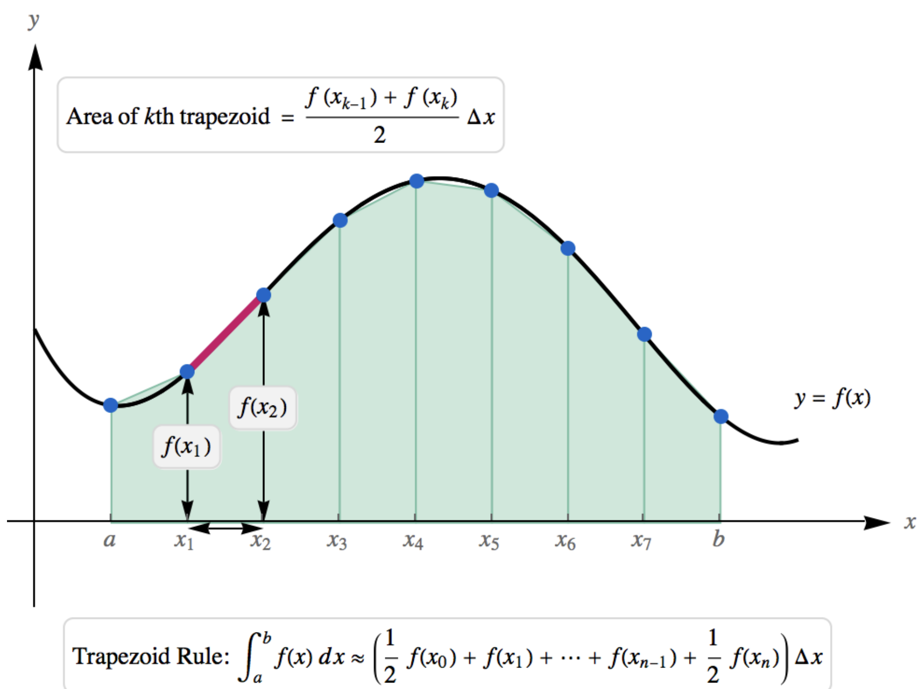
Definition 6. *Suppose f is defined and integrable on $[a, b]$. The Midpoint Rule approximation to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is*

$$M(n) = \sum_{k=1}^{n} f(m_k)\Delta x,$$

where $\Delta x = (b-a)/n$, $x_k = a + k\Delta x$, and $m_k = (x_{k-1} + x_k)/2$ is the midpoint of $[x_{k-1}, x_k]$ for $k = 1, \ldots, n$.

Example 7 (§7.7 Ex 12)**.** *Find the Midpoint Rule approximations to $\int_1^9 x^3\,dx$ using $n = 1, 2,$ and 4 subintervals.*

The Trapezoid Rule approximates the region under the curve by trapezoids.



The bases of the trapezoids have length Δx. The sides of the kth trapezoid have lengths $f(x_{k-1})$ and $f(x_k)$ for $k = 1, \dots n$. Therefore, the net area of the kth trapezoid is $\left(\frac{f(x_{k-1})+f(x_k)}{2}\right)\Delta x$.

Definition 8. *Suppose f is defined and integrable on $[a, b]$. The Trapezoid Rule approximation to $\int_a^b f(x)dx$ using n equally spaced subintervals on $[a, b]$ is*

$$T(n) = \left(\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right)\Delta x,$$

where $\Delta x = (b-a)/n$ and $x_k = a + k\Delta x$ for $k = 0, 1, \dots, n$.

Note that we have the identity

$$T(2n) = \frac{T(n) + M(n)}{2}.$$

Example 9 (§7.7 Ex 16)**.** *Find the Trapezoid Rule approximations to $\int_1^9 x^3\,dx$ using $n = 2, 4$, and 8 subintervals.*

An improvement over the Midpoint Rule and the Trapezoid Rule results when the graph of f is approximated with curves rather than line segments. Suppose we use three neighboring points on the curve $y = f(x)$, say $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))$. These three points determine a parabola, and we can find the net area under the parabola. Simpson's Rule computes the net area in this way:

Definition 10. *Suppose f is defined and integrable on $[a, b]$ and $n \geq 2$ is an even integer. The Simpson's Rule approximation to $\int_a^b f(x)dx$ using n equally spaced subintervals on $[a, b]$ is*

$$S(n) = (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n))\frac{\Delta x}{3},$$

where n is an even integer, $\Delta x = (b-a)/n$, and $x_k = a + k\Delta x$, for $k = 0, 1, \dots, n$.