Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Numerical integration, wrap up

Briggs-Cochran-Gillett §7.7 pp. 557 - 566

An improvement over the Midpoint Rule and the Trapezoid Rule results when the graph of f is approximated with curves rather than line segments. Suppose we use three neighboring points on the curve $y = f(x)$, say $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))$. These three points determine a parabola, and we can find the net area under the parabola.

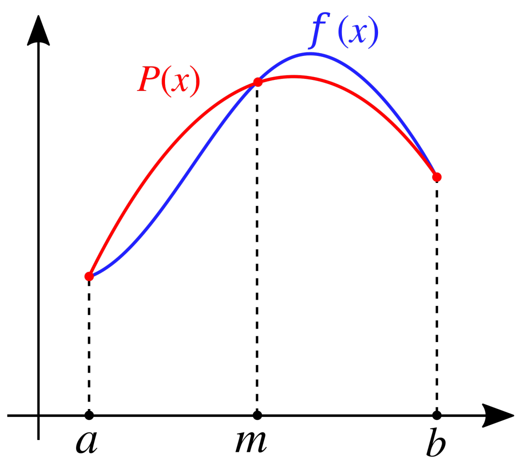


Image credit: Wikipedia, Popletibus, CC BY-SA 4.0

Simpson's Rule computes the net area in this way:

Definition 1. *Suppose f is defined and integrable on $[a, b]$ and $n \geq 2$ is an even integer. The Simpson's Rule approximation to $\int_a^b f(x)dx$ using n equally spaced subintervals on $[a, b]$ is*

$$S(n) = (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n))\frac{\Delta x}{3},$$

where n is an even integer, $\Delta x = (b - a)/n$, and $x_k = a + k\Delta x$, for $k = 0, 1, \ldots, n$.

Note that apart from the first and last terms, the coefficients alternate between 4 and 2; *n must be an even integer* for this rule to apply.

Moreover, note that we have the following relationship among the Midpoint Rule, the Trapezoid Rule, and Simpson's Rule:

$$S(2n) = \frac{2M(n) + T(n)}{3}.$$

Example 2 (§7.7 Ex 39)**.** *Use Simpson's Rule with $n = 4$ to compute an estimate to*

$$\int_0^4 (3x^5 - 8x^3)\,dx.$$

How good are these numerical approximations? Typically the Midpoint Rule is twice as accurate as the Trapezoid Rule, and Simpson's Rule is more accurate than the former two.

Theorem 3. *Assume that f'' is continuous on the interval $[a, b]$ and that k is a bound on the absolute value of the second derivative of f on $[a, b]$. That is $|f''(x)| \leq k$ for all x in $[a, b]$. Then we have the following:*

$$E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 \qquad E_T \leq \frac{k(b-a)}{12}(\Delta x)^2,$$

where E_M is the error involved in estimating the integral with the Midpoint Rule and E_T is the error involved in estimating the integral with the Trapezoid Rule.

Here is the error estimate for Simpson's Rule:

Theorem 4. *Assume that $f^{(4)}$ is continuous on the interval $[a, b]$ and that K is a bound on the absolute value of the fourth derivative of f on $[a, b]$. That is, $|f^{(4)}(x)| \leq K$ for all x in $[a, b]$. Then we have*

$$E_S \leq \frac{K(b-a)}{180}(\Delta x)^4.$$

Note that the error estimates for the Midpoint Rule and the Trapezoid Rule are second order and Simpson's Rule is fourth order.

2 Improper integrals

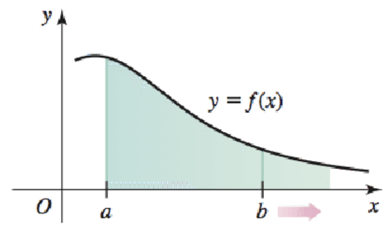
Briggs-Cochran-Gillett §7.8 pp. 570 - 578

By an *improper integral* we refer to an integral where

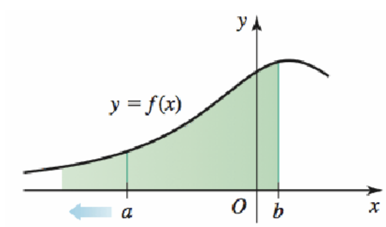
- the interval of integration is infinite, or
- the integrand is unbounded on the interval of integration.

DEFINITION **Improper Integrals over Infinite Intervals**

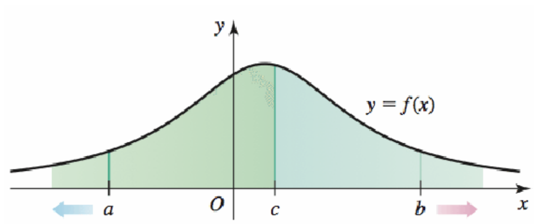
1. If f is continuous on $[a, \infty)$, then $\displaystyle\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$.



2. If f is continuous on $(-\infty, b]$, then $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$.



3. If f is continuous on $(-\infty, \infty)$, then $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx$ where c is any real number.



If the limits in cases 1–3 exist, the improper integral is said to **converge**; otherwise, they **diverge**.

Example 5 (§7.8 Ex 6)**.** *Evaluate the integral* $\int_0^\infty \frac{dx}{(x+1)^3}$ *or state that it diverges.*

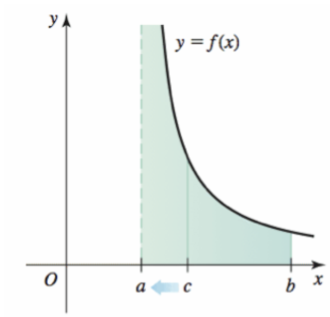
Example 6 (§7.8 Ex 13). *Evaluate the integral* $\int_0^\infty e^{-ax}dx, a > 0$ *or state that it diverges.*

Example 7 (§7.8 Ex 16). *Evaluate the integral* $\int_0^\infty \frac{p}{\sqrt[5]{p^2+1}}dp$ *or state that it diverges.*

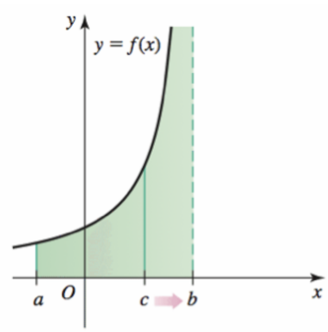
Example 8 (§7.8 Ex 30). *Find the volume (if possible) of the solid of revolution given by rotating the region bounded by* $f(x) = (x^2+1)^{-1/2}$ *and the* x*-axis on the interval* $[2, \infty)$ *about the* x*-axis.*

DEFINITION **Improper Integrals with an Unbounded Integrand**

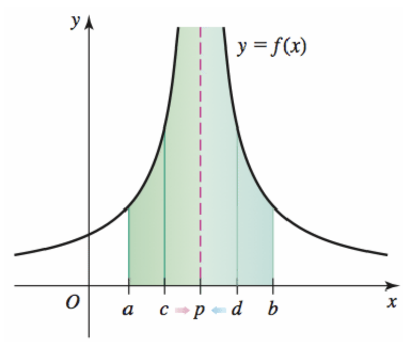
1. Suppose f is continuous on $(a, b]$ with $\lim_{x \to a^+} f(x) = \pm\infty$. Then, $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$.



2. Suppose f is continuous on $[a, b)$ with $\lim_{x \to b^-} f(x) = \pm\infty$. Then, $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$.



3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then, $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$ where the integrals on the right side are evaluated as improper integrals.



If the limits in cases 1-3 exists, the improper integrals **converge**; otherwise they **diverge**.

Example 9 (§7.8 Ex 44). *Evaluate the integral* $\int_1^\infty \frac{dx}{\sqrt[3]{x-1}}$ *or state that it diverges.*