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What is on today

1 Improper integrals wrap up

Briggs-Cochran-Gillett §7.8 pp. 570 - 578

Example 1 (§7.8 Ex 44)**.** *Evaluate the integral* $\int_1^\infty \frac{dx}{\sqrt[3]{x-1}}$ *or state that it diverges.*

2 Sequences and series: an overview

Briggs-Cochran-Gillett §8.1 pp. 596 - 604

A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, \}.$$

A sequence may be generated by a recurrence relation of the form $a_{n+1} = f(a_n)$ for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an explicit formula of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Example 2 (§8.1 Ex 21). *Write the first four terms of the sequence* $\{a_n\}$ *defined by the recurrence relation* $a_{n+1} = 3a_n^2 + n + 1; a_1 = 0$.

Perhaps the most important question about a sequence is this: if you go father and farther out in the sequence $a_{100}, \ldots, a_{100000}, \ldots, a_{10000000000}, \ldots$, how do the terms of the sequence behave? Is there a limiting value, or do they grow without bound?

Definition 3. *If the terms of a sequence* $\{a_n\}$ *approach a unique number* L *as* n *increases – that is, if* a_n *can be made arbitrarily close to* L *by taking* n *sufficiently large – then we say* $\lim_{n\to\infty} a_n = L$ *exists, and the sequence converges to* L*. If the terms of the sequence do not approach a single number as* n *increases, the sequence has no limit, and the sequence diverges.*

Given a sequence $\{a_1, a_2, a_3, \ldots\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an infinite series. The sequence of partial sums $\{S_n\}$ associated with this series has the terms

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series converges to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also diverges.

Example 4 (§8.1 Ex 61)**.** *For the infinite series* $4 + 0.9 + 0.09 + 0.009 + \cdots$, *find the first four terms of the sequence of partial sums. Then make a conjecture about the value of the infinite series.*

Example 5 (§8.1 Ex 73)**.** *Consider the infinite series* $\sum_{k=1}^{\infty} 3^{-k}$. *Write out the first four terms of the sequence of partial sums. Estimate the limit of* $\{S_n\}$ *or state that it does not exist.*

Example 6 (§8.1 Ex 74)**.** *Consider the infinite series* $\sum_{k=1}^{\infty} k$. *Write out the first four terms of the sequence of partial sums. Estimate the limit of* $\{S_n\}$ *or state that it does not exist.*

3 Sequences

Briggs-Cochran-Gillett §8.2 pp. 607 - 611

A fundamental question about sequences concerns the behavior of the terms as we go out farther and farther in the sequence. Below we state a few theorems regarding limits of sequences:

Theorem 7 (Limits of sequences from limits of functions)**.** *Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.*

Theorem 8 (Limit laws for sequences)**.** *Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then*

1. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} ca_n = cA$, *where c is a real number*
3. $\lim_{n\to\infty} a_n b_n = AB$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, *provided $B \neq 0$*

Example 9 (§8.2 Ex 10)**.** *Find the limit of the sequence $\{\frac{n^{12}}{3n^{12}+4}\}$ or determine that the limit does not exist.*

Example 10 (§8.2 Ex 22)**.** *Find the limit of the sequence $\{(1+\frac{4}{n})^{3n}\}$ or determine that the limit does not exist.*