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What is on today

1 Sequences

Briggs-Cochran-Gillett §8.2 pp. 607 - 611

We can use earlier results on growth rates of functions (§4.7) to compare growth rates of sequences:

Theorem 1 (Growth Rates of Sequences)**.** *The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:*

$$\{(\ln n)^q\} \ll \{n^p\} \ll \{n^p(\ln n)^r\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The order applies for positive real numbers p, q, r, s and $b > 1$.

Example 2 (§8.2 Ex 67)**.** *Use the previous theorem to find the limit of the sequence $\left\{\frac{n^{1000}}{2^n}\right\}$ or state that it diverges.*

2 Infinite series

Briggs-Cochran-Gillett §8.3 pp. 619 - 623

We will focus our attention on two types of infinite series: geometric series and telescoping series.

As a preliminary step to geometric series, we discuss geometric sums. A geometric sum with n terms has the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k,$$

where $a \neq 0$ and r are real numbers. r is called the ratio of the sum. We can compute the value of the geometric sum

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \tag{1}$$

by doing the following: multiply both sides of (1) by r:

$$rS_n = ar + ar^2 + \cdots + ar^n$$

and consider the difference of the above with (1):

$$S_n - rS_n = a - ar^n.$$

Then solving for S_n gives that

$$S_n = a\frac{1-r^n}{1-r}.$$

Example 3 (§8.3 Ex 7)**.** *Compute* $\sum_{k=0}^{8} 3^k$.

Example 4 (§8.3 Ex 15)**.** *Compute* $\sum_{k=0}^{20} (-1)^k$.

Now we consider geometric series. The geometric sums $S_n = \sum_{k=0}^{n-1} ar^k$ form the sequence of partial sums for the geometric series $\sum_{k=0}^{\infty} ar^k$. The value of the geometric series is the limit of its sequence of partial sums. Thus we have

$$\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} \sum_{k=0}^{n-1} ar^k = \lim_{n\to\infty} a\frac{1-r^n}{1-r}.$$

Then using a theorem from the last class, that

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \le -1 \text{ or } r > 1, \end{cases}$$

we find that

Theorem 5. *Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges.*

Example 6 (§8.3 Ex 19, 24, 26, 40)**.** *Evaluate each geometric series or state that it diverges.*

1. $\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k$.

2. $1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \cdots$

3. $\sum_{m=2}^{\infty} \frac{5}{2^m}$.

4. $\sum_{k=1}^{\infty} 3\left(-\frac{1}{8}\right)^{3k}$

Example 7 (§8.3 Ex 46)**.** *Write the repeating decimal* $0.\overline{27} = .272727\cdots$ *as a geometric series and then as a rational number.*

We were able to compute geometric series by finding a formula for the sequence of partial sums and then evaluating the limit of the sequence. Not many infinite series can be computed in this way. However, for another class of series, called telescoping series, we can also do something similar. Here is an example.

Example 8 (§8.3 Ex 56)**.** *Consider the telescoping series* $\sum_{k=1}^{\infty}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)$. *Find a formula for the nth term of the sequence of partial sums* S_n*. Then evaluate* $\lim_{n\to\infty} S_n$ *to obtain the value of the series or state that the series diverges.*