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What is on today

1 The divergence and integral tests

Briggs-Cochran-Gillett §8.4 pp. 627 - 638

As we saw in the last class, for geometric series and telescoping series, the sequence of partial sums can be found and its limit evaluated. But actually, it is difficult or impossible to find an explicit formula for the sequence of partial sums for most infinite series. So it is tough to get the value of most convergent series.

So we try to answer another question: given an infinite series, does it converge? If the answer is no, the series diverges and there are no further questions to ask. But if the answer is yes, the series converges, and it may be possible to estimate its value.

We give a criterion for when an infinite series diverges:

Theorem 1 (Divergence Test)**.** *If* $\sum a_k$ *converges, then* $\lim_{k\to\infty} a_k = 0$*. Equivalently, if* $\lim_{k\to\infty} a_k \neq 0$*, then the series diverges.*

Note that the Divergence Test CANNOT be used to conclude that a series converges.

Example 2 (§8.4 Ex 9, 10, 12)**.** *Use the Divergence Test to determine whether each of the series below diverges or state that the test is inconclusive.*

1. $\sum_{k=0}^{\infty} \frac{k}{2k+1}$

2. $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

3. $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

Our next test comes via a question about the *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

Does it converge? If we look at the sequence of partial sums, we have

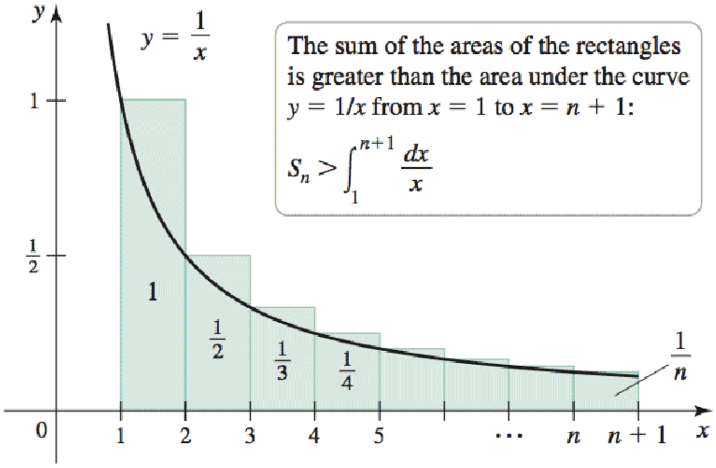
$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}, \end{aligned}$$

and there's no obvious pattern in this sequence. In fact, there no simple explicit formula for the S_n.

How do we make sense of the S_n? Observe that

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is the result of computing a left Riemann sum of the function $y = \frac{1}{x}$ on the interval $[1, n+1]$:



Comparing the sum of the areas of the n rectangles with the area under the curve, we see that

$$S_n > \int_1^{n+1} \frac{dx}{x}.$$

We know that

$$\int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$

increases without bound as n increases. Thus S_n also increases without bound, and the harmonic series diverges.

Theorem 3. *The harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ *diverges, even though the terms of the series approach zero.*

The ideas used to prove that the harmonic series diverges are now used to prove a new convergence test, the Integral Test. This test applies only to series with positive terms.

Theorem 4 (Integral Test)**.** *Suppose f is a* ***continuous, positive, decreasing*** *function, for $x \geq 1$, and let $a_k = f(k)$ for $k = 1, 2, 3, \ldots$. Then*

$$\sum_{k=1}^{\infty} a_k \qquad \text{and} \qquad \int_1^{\infty} f(x)dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is not equal to the value of the series.

Example 5 (§8.4 Ex 20, 21)**.** *Use the Integral Test to determine the convergence or divergence of the following series, or state that the test does not apply.*

1. $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$

2. $\sum_{k=1}^{\infty} ke^{-2k^2}$

The integral test is used to prove the following:

Theorem 6 (p-series test)**.** *The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.*

Example 7 (§8.4 Ex 30, 34)**.** *Determine the convergence or divergence of the following series:*

1. $\sum_{k=2}^{\infty} \frac{k^e}{k^\pi}$

2. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{27k^2}}$

Here are some useful properties of convergent series:

Theorem 8 (Properties of convergent series)**.**

1. *Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c \sum a_k = cA$.*
2. *Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges, and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.*
3. *If M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ either both converge or diverge.*

Example 9 (§8.4 Ex 46)**.** *Use the properties of infinite series to evaluate $\sum_{k=1}^{\infty} \left(2\left(\frac{3}{5}\right)^k + 3\left(\frac{4}{9}\right)^k\right)$.*

Example 10 (§8.4 Ex 54, 56). *Determine whether the following series converge or diverge:*

1. $\sum_{k=0}^{\infty} \frac{10}{k^2+9}$

2. $\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$.