Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 The divergence and integral tests

Briggs-Cochran-Gillett §8.4 pp. 627 - 638

We begin by reviewing some material from right before spring break.

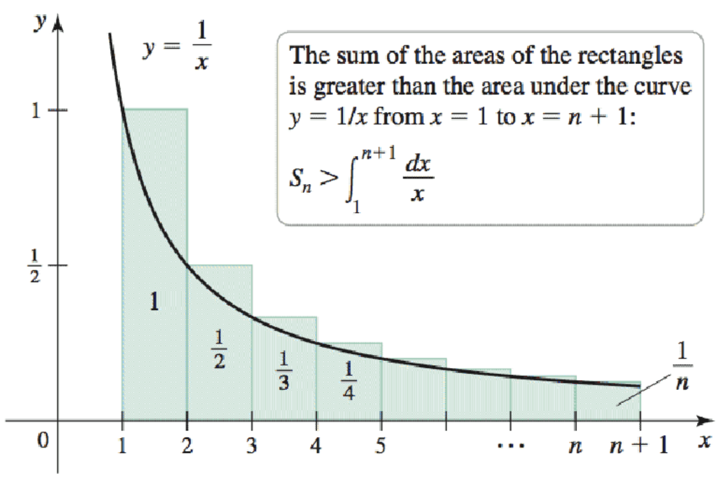
Consider the *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

We investigate whether it converges. Observe that

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is the result of computing a left Riemann sum of the function $y = \frac{1}{x}$ on the interval $[1, n+1]$:



Comparing the sum of the areas of the n rectangles with the area under the curve, we see that

$$S_n > \int_1^{n+1} \frac{dx}{x}.$$

We know that

$$\int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$

increases without bound as n increases. Thus S_n also increases without bound, and the harmonic series diverges.

Theorem 1. *The harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ *diverges, even though the terms of the series approach zero.*

The ideas used to prove that the harmonic series diverges are now used to prove a new convergence test, the Integral Test. This test applies only to series with positive terms.

Theorem 2 (Integral Test)**.** *Suppose f is a* ***continuous, positive, decreasing*** *function, for $x \geq 1$, and let $a_k = f(k)$ for $k = 1, 2, 3, \ldots$. Then*

$$\sum_{k=1}^{\infty} a_k \qquad \textit{and} \qquad \int_1^{\infty} f(x)dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is not equal to the value of the series.

The integral test is used to prove the following:

Theorem 3 (p-series test)**.** *The p-series* $\sum_{k=1}^{\infty} \frac{1}{k^p}$ *converges for $p > 1$ and diverges for $p \leq 1$.*

Example 4 (§8.4 Ex 54, 56)**.** *Determine whether the following series converge or diverge:*

1. $\sum_{k=0}^{\infty} \frac{10}{k^2+9}$

2. $\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$.

Theorem 5 (Estimating series with positive terms)**.** *Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series, and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies*

$$R_n < \int_n^\infty f(x)dx.$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^\infty f(x)dx < \sum_{k=1}^\infty a_k < S_n + \int_n^\infty f(x)dx.$$

Example 6 (§8.4 Ex 41)**.** *Consider the convergent series* $\sum_{k=1}^\infty \frac{1}{k^3}$.

1. *Find an upper bound for the remainder in terms of* n.

2. *Find how many terms are needed to ensure the remainder is less than* 10^{-3}.

3. *Find lower and upper bounds on the exact value of the series.*

4. *Find an interval in which the value of the series must lie if you approximate it using ten terms of the series.*