Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 The comparison, limit comparison, ratio, and root tests

Briggs-Cochran-Gillett §8.5 pp. 641 - 647

The Comparison Test lets us leverage information about the convergence or divergence of a series by comparing it to the behavior of another series:

Theorem 1 (Comparison Test)**.** *Let $\sum a_k$ and $\sum b_k$ be series with positive terms.*

1. *If $0 < a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
2. *If $0 < b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Example 2 (§8.5 Ex 27, 38, 54)**.** *Use the Comparison Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$

2. $\sum_{k=2}^{\infty} \frac{1}{(k \ln k)^2}$

3. $\sum_{k=2}^{\infty} \frac{5 \ln k}{k}$

The Comparison Test should be tried if there is an obvious comparison series and the necessary inequality is easily established. But sometimes the right inequality is not easy to establish. In this case, it is often easier to use a more refined test called the Limit Comparison Test:

Theorem 3 (Limit Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.*

1. *If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.*
2. *If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
3. *If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Example 4 (§8.5 Ex 30, 34, 46). *Use the Limit Comparison Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{0.0001}{k+4}$

2. $\sum_{k=1}^{\infty} \frac{1}{3^k - 2^k}$

3. $\sum_{k=1}^{\infty} \frac{2^k}{e^k - 1}$

Here is another test for convergence:

Theorem 5 (Ratio Test)**.** *Let* $\sum a_k$ *be an infinite series with positive terms and let* $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

1. *If* $0 \leq r < 1$*, the series converges.*
2. *If* $r > 1$ *(including* $r = \infty$*), the series diverges.*
3. *If* $r = 1$*, the test is inconclusive.*

Example 6 (§8.5 Ex 10, 12)**.** *Use the Ratio Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

2. $\sum_{k=1}^{\infty} \frac{k^k}{2^k}$

Occasionally a series arises for which the preceding tests are difficult to apply. In these situations, we try the Root Test:

Theorem 7 (Root Test)**.** *Let* $\sum a_k$ *be an infinite series with nonnegative terms and let* $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$.

1. *If* $0 \le \rho < 1$, *the series converges.*
2. *If* $\rho > 1$ *(including* $\rho = \infty$*), the series diverges.*
3. *If* $\rho = 1$, *the test is inconclusive.*

Example 8 (§8.5 Ex 20, 25)**.** *Use the Root Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \left(\frac{2k}{k+1}\right)^k$

2. $1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{4}\right)^4 + \cdots$