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What is on today

1 The comparison, limit comparison, ratio, and root tests: wrap up

Briggs-Cochran-Gillett §8.5 pp. 641 - 647

Occasionally a series arises for which the Comparison Test, Limit Comparison Test, and Ratio Tests are difficult to apply. In these situations, we try the Root Test:

Theorem 1 (Root Test)**.** *Let $\sum a_k$ be an infinite series with nonnegative terms and let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$.*

1. *If $0 \le \rho < 1$, the series converges.*
2. *If $\rho > 1$ (including $\rho = \infty$), the series diverges.*
3. *If $\rho = 1$, the test is inconclusive.*

Example 2 (§8.5 Ex 20, 25)**.** *Use the Root Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \left(\frac{2k}{k+1}\right)^k$

2. $1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{4}\right)^4 + \cdots$

Example 3 (§8.5 Ex 44, 47, 52, 57, 58)**.** *Use the test of your choice to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

2. $\sum_{k=1}^{\infty} \frac{k^2+2k+1}{3k^2+1}$

3. $\sum_{k=1}^{\infty} \frac{(k!)^3}{(3k)!}$

4. $\sum_{k=1}^{\infty} \frac{k^8}{k^{11}+3}$

5. $\sum_{k=1}^{\infty} \frac{1}{(1+p)^k}, p > 0$

2 Alternating series

Briggs-Cochran-Gillett §8.6 pp. 649 - 652

The previous tests focused on infinite series with positive terms. We shift our attention to studying series with terms that have strictly alternating signs, as in the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots .$$

The factor $(-1)^{k+1}$ (or possibly $(-1)^k$) provides the alternating signs.

Theorem 4 (Alternating Series Test)**.** *The alternating series* $\sum(-1)^{k+1}a_k$ *converges if*

1. *the terms of the series are nonincreasing in magnitude (*$0 < a_{k+1} \leq a_k$*, for* k *greater than some index* N*) and*
2. $\lim_{k\to\infty} a_k = 0$.

What does the Alternating Series Test tell us about the alternating harmonic series?

Theorem 5. *The alternating harmonic series* $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ *converges.*

For series of **positive** terms, $\lim_{k\to\infty} a_k = 0$ does **NOT** imply convergence. For **alternating series with nonincreasing** terms, $\lim_{k\to\infty} a_k = 0$ **DOES** imply convergence.

Example 6 (§8.6 Ex 16, 20, 24)**.** *Determine whether the following series converge.*

1. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+10}$
2. $\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k$
3. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln^2 k}$