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What is on today

1 Alternating series

Briggs-Cochran-Gillett §8.6 pp. 649 - 656

We begin by reviewing the Alternating Series Test:

Theorem 1 (Alternating Series Test)**.** *The alternating series* $\sum(-1)^{k+1}a_k$ *converges if*

1. *the terms of the series are nonincreasing in magnitude* ($0 < a_{k+1} \le a_k$, *for* k *greater than some index* N) *and*
2. $\lim_{k\to\infty} a_k = 0$.

For series of **positive** terms, $\lim_{k\to\infty} a_k = 0$ does **NOT** imply convergence. For **alternating series with nonincreasing** terms, $\lim_{k\to\infty} a_k = 0$ **DOES** imply convergence.

Example 2 (§8.6 Ex 16, 20, 24)**.** *Determine whether the following series converge.*

1. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+10}$
2. $\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k$
3. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln^2 k}$

Recall that if a series converges to a value S, then the remainder is $R_n = S - S_n$, where S_n is the sum of the first n terms of the series. An upper bound on the magnitude of the remainder (the *absolute error*) in an alternating series arises form the following observation: when the terms are nonincreasing in magnitude, the value of the series is always trapped between successive terms of the sequence of partial sums. Thus we have

$$|R_n| = |S - S_n| \leq |S_{n+1} - S_n| = a_{n+1}.$$

This justifies the following theorem:

Theorem 3 (Remainder in Alternating Series)**.** *Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder. Then $|R_n| \leq a_{n+1}$.*

Example 4 (§8.6 Ex 31)**.** *Determine how many terms of the convergent series $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$ must be summed to be sure that the remainder is less than 10^{-4} in magnitude. (Although you do not need it, the exact value is $\pi/4$.)*

Now we will consider infinite series $\sum a_k$ where the terms are allowed to be any real numbers (not just all positive or alternating). We first introduce some terminology:

Definition 5. *If $\sum |a_k|$ converges, then we say that $\sum a_k$ converges absolutely. If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ converges conditionally.*

The series $\sum \frac{(-1)^{k+1}}{k^2}$ is an example of an absolutely convergent series because the series of absolute values $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is a convergent p-series. On the other hand, the alternating harmonic series $\sum \frac{(-1)^{k+1}}{k}$ is an example of a conditionally convergent series since the series of absolute values $\sum_{k=1}^{\infty}\frac{1}{k}$ is the harmonic series, which diverges.

Theorem 6. *If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.*

Example 7 (§8.6 Ex 46, 48, 51)**.** *Determine whether the following series converge absolutely, converge conditionally, or diverge.*

1. $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$

2. $\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k$

3. $\sum_{k=1}^{\infty}(-1)^k \tan^{-1} k$

Example 8 (§8.R Ex 28, 30, 36). *Determine whether the following series converge or diverge.*

1. $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}\sqrt{k+1}}$

2. $\sum_{k=1}^{\infty} k\sin\frac{1}{k}$

3. $\sum_{k=1}^{\infty} ke^{-k}$

Table 8.4 Special Series and Convergence Tests

Series or test	Form of series	Condition for convergence	Condition for divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} a\,r^k,\ a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} a\,r^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$, where $a_k \geq 0$	$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$	$\lim_{k\to\infty} \sqrt[k]{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	$0 < b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0,\ b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0,\ 0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k = 0$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k,\ a_k$ arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$		Applies to arbitrary series