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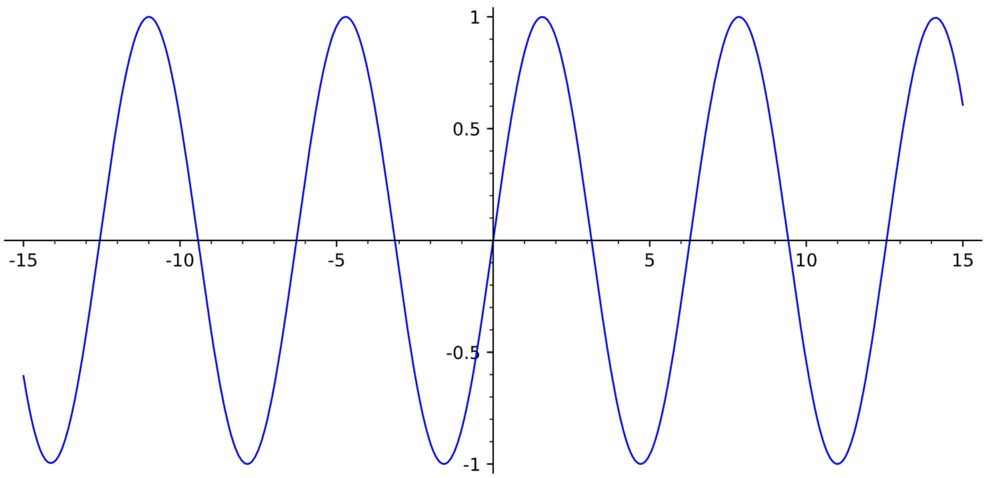
What is on today

1 Approximating functions with polynomials

Briggs-Cochran-Gillett §9.1 pp. 661 - 667

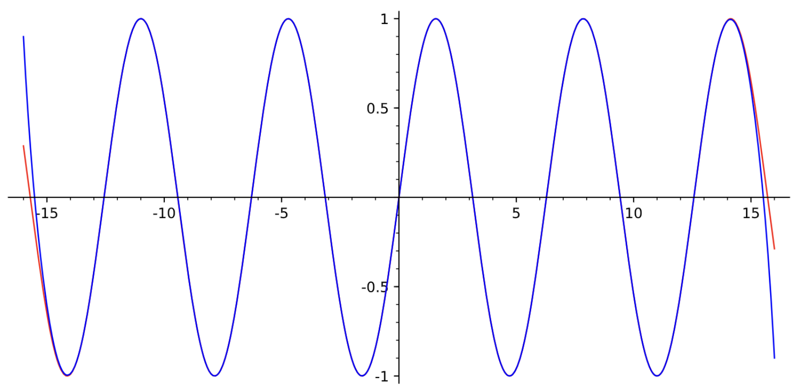
Guess the function!



It's the graph of

$$\begin{aligned}
y = &-\frac{1}{20397882081197443358640281739902897356800000000}x^{39} + \frac{1}{13763753091226345046315979581580902400000000}x^{37} \\
&- \frac{1}{10333147966386144929666651337523200000000}x^{35} + \frac{1}{8683317618811886495518194401280000000}x^{33} \\
&- \frac{1}{8222838654177922817725562880000000}x^{31} + \frac{1}{8841761993739701954543616000000}x^{29} - \frac{1}{10888869450418352160768000000}x^{27} \\
&+ \frac{1}{15511210043330985984000000}x^{25} - \frac{1}{25852016738884976640000}x^{23} + \frac{1}{51090942171709440000}x^{21} - \frac{1}{121645100408832000}x^{19} \\
&+ \frac{1}{355687428096000}x^{17} - \frac{1}{1307674368000}x^{15} + \frac{1}{6227020800}x^{13} - \frac{1}{39916800}x^{11} + \frac{1}{362880}x^{9} - \frac{1}{5040}x^{7} + \frac{1}{120}x^{5} - \frac{1}{6}x^{3} + x
\end{aligned}$$

Why does it look like the graph of $y = \sin x$?



($y = \sin x$ is in red)

This is the sort of thing we will investigate today. First, a *power series* is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots ,$$

or more generally,

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + \cdots ,$$

where the center of the series a and the coefficients c_k are constants. Another way of thinking about this is that a power series is built up from polynomials of increasing degree:

$$\begin{aligned}
&c_0 \\
&c_0 + c_1 x \\
&c_0 + c_1 x + c_2 x^2 \\
&\vdots \\
&c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n = \sum_{k=0}^{n} c_k x^k \\
&\vdots \\
&c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n + \cdots = \sum_{k=0}^{\infty} c_k x^k .
\end{aligned}$$

With this perspective, we begin our exploration of power series by using polynomials to approximate functions.

Earlier, we learned that if a function f is differentiable at a point a, then it can be approximated near a by its tangent line, which is the linear approximation to f at the point a. The linear approximation at a is given by

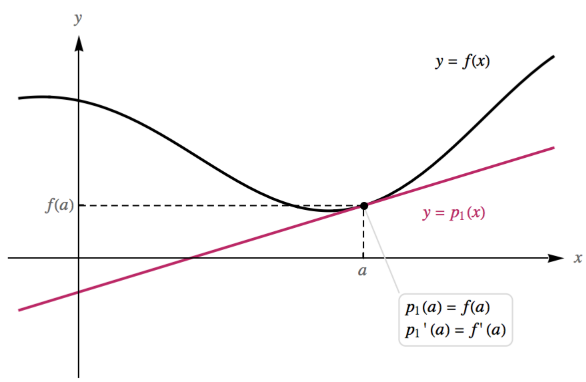
$$y = f(a) + f'(a)(x - a).$$

Because the linear approximation is a first-degree polynomial, we name it p_1:

$$p_1(x) := f(a) + f'(a)(x - a).$$

It matches f in value and in slope at a:

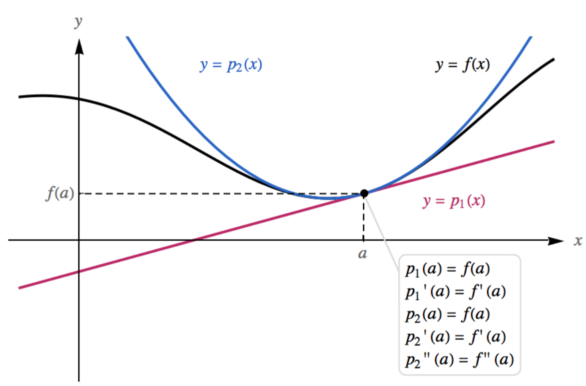
$$p_1(a) = f(a), p_1'(a) = f'(a).$$



Linear approximation works well if f has a fairly constant slope near a. However, if f has a lot of curvature near a, then the tangent line may not provide an accurate approximation. To remedy this situation, we create a quadratic approximating polynomial by adding one new term to the linear polynomial. Denoting this new polynomial p_2, we let

$$p_2(x) := f(a) + f'(a)(x - a) + c_2(x - a)^2 = p_1(x) + c_2(x - a)^2.$$

To determine c_2, we require that p_2 agree with f in value, slope, and concavity at a:



By construction, it already agrees with f in value and slope at a. Differentiating $p_2(x)$ twice and substituting $x = a$ yields

$$p_2''(a) = 2c_2 = f''(a).$$

Thus it follows that $c_2 = \frac{1}{2}f''(a)$ and

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Example 1 (§9.1 Ex 8, 10)**.**

(a) Find the linear approximating polynomial for the following functions centered at the given point a.

(b) Find the quadratic approximating polynomial for the following functions centered at the given point a.

(c) Use the polynomials obtained in the first two parts to approximate the given quantity.

1. $f(x) = \frac{1}{x}, a = 1,$ *approximate* $\frac{1}{1.05}$

2. $f(x) = \sqrt{x}, a = 4,$ *approximate* $\sqrt{3.9}$

Assume that f and its first n derivatives exist at a. Our goal is to find an nth-degree polynomial that approximates the values of f near a. The first step is to use p_2 to obtain a cubic polynomial p_3 of the form

$$p_3(x) = p_2(x) + c_3(x-a)^3$$

that satisfies the four matching conditions

$$p_3(a) = f(a), p_3'(a) = f'(a), p_3''(a) = f''(a), p_3'''(a) = f'''(a).$$

Because p_3 is built using p_2, the first three conditions are met. The last condition is used to determine c_3. Differentiating as before, we find $p_3'''(x) = 3 \cdot 2c_3 = 3!c_3$, or in other words,

$$c_3 = \frac{f'''(a)}{3!}.$$

Continuing in this way, building each new polynomial on the previous polynomial, we construct the Taylor polynomials:

Definition 2. *Let f be a function with $f', f'', \ldots$ and $f^{(n)}$ defined at a. The nth order Taylor polynomial for f with its center at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a. That is,*

$$p_n(a) = f(a), p_n'(a) = f'(a), \ldots, p_n^{(n)}(a) = f^{(n)}(a).$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example 3 (§9.1 Ex 15)**.** *Let $f(x) = \cos x$. Find the nth-order Taylor polynomials of $f(x)$ centered at 0 for $n = 0, 1, 2$.*

Example 4 (§9.1 Ex 32)**.** *Let $f(x) = \cos x$ and $a = \pi/6$. Find the nth-order Taylor polynomials for $f(x)$ centered at a, for $n = 0, 1, 2$.*