Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Approximating functions with polynomials

Briggs-Cochran-Gillett §9.1 pp. 665 - 671

Today we look at approximations obtained with Taylor polynomials and give estimates on the remainder in a Taylor polynomial.

Example 1. *Find the Taylor polynomials of order $n = 0, 1, 2, 3$ for $f(x) = e^x$ centered at 0. Use the polynomials to approximate $e^{0.1}$. Find the absolute errors $|f(x) - p_n(x)|$ in the approximations. (Hint: $e^{0.1} \approx 1.1051709$.)*

Example 2. *Use a Taylor polynomial of order 2 to approximate $\sqrt{18}$.*

Taylor polynomials provide good approximations to functions near a specific point. But how accurate are the approximations? To answer this, we define the **remainder** in a Taylor polynomial:

Definition 3. *Let p_n be the Taylor polynomial of order n for f. The remainder in using p_n to approximate f at the point x is $R_n(x) = f(x) - p_n(x)$.*

We have the following result quantifying the remainder:

Theorem 4 (Taylor's theorem)**.** *Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I, we have $f(x) = p_n(x) + R_n(x)$, where p_n is the nth-order Taylor polynomial for f centered at a and the remainder is*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some point c between x and a.

Example 5 (§9.1 Ex 51, 52)**.** *Find the remainder R_n for the nth order Taylor polynomial centered at a for the given functions. Express the result for a general value of n.*

1. $f(x) = e^{-x}, a = 0$.

2. $f(x) = \cos x, a = \frac{\pi}{2}$.

The difficulty in estimating the remainder is finding a bound for $|f^{(n+1)}(c)|$. Assuming this can be done, we have the following theorem:

Theorem 6 (Estimate of the remainder)**.** *Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \le M$, for all c between a and x, inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies*

$$|R_n(x)| = |f(x) - p_n(x)| \le M\frac{|x-a|^{n+1}}{(n+1)!}.$$

Example 7 (§9.1 Ex 57)**.** *Use the remainder to find a bound on the error in approximating* $e^{0.25}$ *with the 4th-order Taylor polynomial centered at 0.*

Example 8 (§9.1 Ex 66)**.** *Consider the approximation* $\sqrt{1+x} \approx 1 + \frac{x}{2}$ *on* $[-0.1, 0.1]$*. Use the remainder to find a bound on the error on the given interval.*

Example 9 (§9.1 Ex 68)**.** *What is the minimum order of the Taylor polynomial required to approximate* $\sin 0.2$ *with an absolute error no greater than* 10^{-3}*?*