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What is on today

1 Taylor series

Briggs-Cochran-Gillett §9.3 pp. 684 - 690

Suppose a function f has derivatives $f^{(k)}(a)$ of all orders at the point a. If we write the nth order Taylor polynomial for f centered at a and allow n to increase indefinitely, we get a power series – the Taylor series for f centered at a:

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots = \sum_{k=0}^{\infty} c_k(x-a)^k,$$

where the coefficients are given by

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad k = 0, 1, 2, \ldots.$$

The special case of a Taylor series centered at 0 is called a Maclaurin series.

For the Taylor series to be useful, we need to know two things: the values of x for which the Taylor series converges, and the values of x for which the Taylor series for f *equals* f. We will study the first issue now in a few examples. We will look at the second issue during the next class.

Example 1 (§9.3 Ex 9, 10)**.** *For each of the following functions,*

(a) Find the first four nonzero terms of the Maclaurin series.

(b) Write the power series using summation notation.

(c) Determine the interval of convergence.

1. $f(x) = e^{-x}$

2. $f(x) = \cos(2x)$

Example 2 (§9.3 Ex 24, 26). *Find the first four nonzero terms of the Taylor series for the given function centered at a and write the power series using summation notation.*

1. $f(x) = 1/x, a = 2$

2. $f(x) = e^x, a = \ln 2$

Here are commonly used Taylor series and the functions to which they converge.

Table 9.5

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \text{ for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \text{ for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \text{ for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \text{ for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \text{ for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \ \binom{p}{0} = 1$$

Example 3 (§9.3 Ex 30, 36). *Use the Taylor series in Table 9.5 to find the first four nonzero terms of the Taylor series for the following functions centered at 0.*

1. $\sin x^2$

2. $x \tan^{-1} x^2$

We know that if p is a positive integer, then $(1+x)^p$ is a polynomial of degree p; we further have the expansion

$$(1+x)^p = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{p}x^p,$$

where

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \quad \binom{p}{0} = 1,$$

for real numbers p and integers $k \geq 1$.

We extend this to functions $f(x) = (1+x)^p$ where p is a nonzero real number:

Theorem 4 (Binomial series)**.** *For real numbers $p \neq 0$, the Taylor series for $f(x) = (1+x)^p$ centered at 0 is the binomial series*

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{p}{k} x^k &= 1 + \sum_{k=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k \\ &= 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots\end{aligned}$$

The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p). If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

Example 5 (§9.3 Ex 41)**.** *Find the first four nonzero terms of the binomial series centered at 0 for $f(x) = \sqrt[4]{1+x}$. Use this to approximate $\sqrt[4]{1.12}$.*