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What is on today

1 Convergence of Taylor series

Briggs-Cochran-Gillett §9.3 pp. 690 - 694

Today we look at when the Taylor series of f actually converges to f on its interval of convergence. Recall that Taylor's Theorem tells us that

$$f(x) = p_n(x) + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

is the remainder, and c is a point between x and a. We see that the remainder $R_n(x) = f(x) - p_n(x)$ measures the difference between f and the approximating Taylor polynomial p_n. When we say that the Taylor series converges to f at a point x, we mean that the value of the Taylor series at x equals $f(x)$: that is, $\lim_{n\to\infty} p_n(x) = f(x)$. We make this precise below.

Theorem 1 (Convergence of Taylor series)**.** *Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f, for all x in I, if and only if $\lim_{n\to\infty} R_n(x) = 0$, for all x in I, where*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is the remainder at x (with c between x and a).

Example 2 (§9.3 Ex 58, 60)**.** *Find the remainder in the Taylor series centered at the point a for the following functions. Then show that $\lim_{n\to\infty} R_n(x) = 0$ for all x in the interval of convergence.*

1. $f(x) = \cos(2x), a = 0$

2. $f(x) = \cos x, a = \pi/2$

2 Working with Taylor series

Briggs-Cochran-Gillett §9.4 pp. 696 - 699

We now know the Taylor series for many familiar functions, and we have a number of new tools for working with power series. Here we wrap up some additional techniques that make use of what we've studied thus far.

Example 3 (§9.4 Ex 7, 10, 14)**.** *Evaluate the following limits using Taylor series.*

1. $\lim_{x\to 0} \frac{e^x - 1}{x}$

2. $\lim_{x\to 0} \frac{\sin 2x}{x}$

3. $\lim_{x\to\infty} x \sin \frac{1}{x}$

Example 4 (§9.4 Ex 31, 32)**.** *For each of the following functions,*

(a) Differentiate the Taylor series about 0 for the following functions.

(b) Identify the function represented by the differentiated series.

(c) Give the interval of convergence of the power series for the derivative.

1. $f(x) = \tan^{-1} x$

2. $f(x) = -\ln(1-x)$