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What is on today

1 Introduction

- Please refer to the syllabus for all course details.
- How lectures will work:
  - Notes will be available online (http://math.bu.edu/people/jbala/124.html) for use in each class.
  - We will use Learning Catalytics (through MyLab Math).
- This course will focus on applications of integration, techniques for integration, sequences and series, and power series (Chapters 6 to 11 of the textbook).

2 Velocity and net change

Briggs-Cochran-Gillett-Schulz §6.1 pp. 403 - 410

Suppose you are driving along a straight highway and your **position** relative to a reference point (or origin) is $s(t)$ for $t \geq 0$. Your displacement over a time interval $[a, b]$ is the change in position $s(b) - s(a)$. Now assume that $v(t)$ is the **velocity** of the object at t. Recall that

$$\mathbf{v}(\mathbf{t}) = \mathbf{s}'(\mathbf{t}),$$

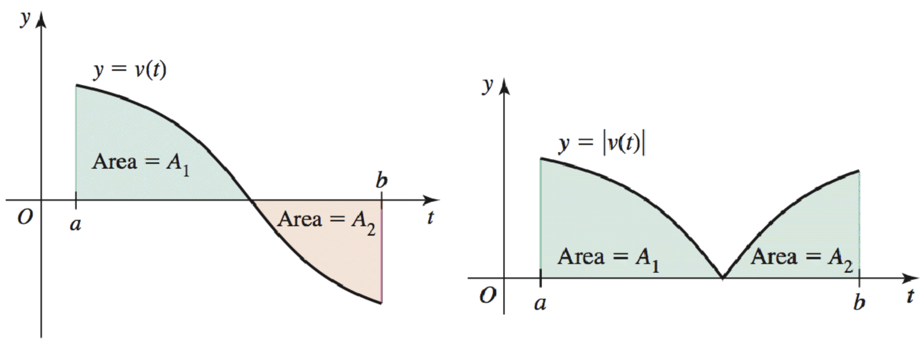
so by the Fundamental Theorem of Calculus (FTC), we have

$$\int_a^b v(t)dt = \int_a^b s'(t)dt = s(b) - s(a) = \textbf{displacement}.$$

We see that the definite integral $\int_a^b v(t)dt$ is the displacement (change in position) between times $t = a$ and $t = b$. Equivalently, the displacement over the time interval $[a, b]$ is the net area under the velocity curve over $[a, b]$.

Not to be confused with displacement is the distance traveled over a time interval, which is the total distance traveled by the object, independent of the direction of motion. If the velocity is positive, the object moves in the positive direction, and the displacement equals the distance traveled. However, if the velocity changes sign, then the displacement and the distance traveled are not generally equal. To compute the distance traveled, we need the magnitude of the velocity $|v(t)|$, which is called **speed**. We have that

$$\textbf{distance traveled} = \int_a^b |v(t)|dt.$$



Displacement $= A_1 - A_2 = \int_a^b v(t)\,dt$

Distance traveled $= A_1 + A_2 = \int_a^b |v(t)|\,dt$

To find the displacement of an object, we do not need to know its initial position. What happens if we are interested in the actual position of the object at a future time? Suppose we know the velocity of an object and its initial position $s(0)$. The FTC gives us the answer:

$$\int_0^t v(x)dx = \int_0^t s'(x)dx = s(x)\big|_0^t = s(t) - s(0).$$

Rearranging this gives that

$$s(t) = s(0) + \int_0^t v(x)dx.$$

Example 1 (§6.1 Ex. 12)**.** *Let* $v(t) = 6 - 2t$ *on* $0 \le t \le 6$*. Assume* t *is time measured in seconds and velocities have units of m/s.*

1. *Graph the velocity function over the given interval. Then determine when the motion is in the positive direction and when it is in the negative direction.*

2. *Find the displacement over the given interval.*

3. *Find the distance traveled over the given interval.*

Example 2 (§6.1 Ex. 24)**.** *A cyclist rides down a long straight road with a velocity (in m/min) given by $v(t) = 400 - 20t$, for $0 \le t \le 10$, where t is measured in minutes.*

1. *How far does the cyclist travel in the first 5 minutes?*
2. *How far does the cyclist travel in the first 10 minutes?*
3. *How far has the cyclist traveled when her velocity is 250 m/min?*

Just like we calculated position from velocity, we can calculate velocity from acceleration. Given the acceleration $a(x)$, we have

$$\int_0^t a(x)dx = \int_0^t v'(x)dx = v(t) - v(0),$$

which gives

$$v(t) = v(0) + \int_0^t a(x)dx.$$

Example 3 (§6.1 Ex. 38)**.** *A car slows down with an acceleration of $a(t) = -15ft/s^2$. Assume that $v(0) = 60ft/s, s(0) = 0$ and t is measured in seconds.*

1. *Determine the position function for $t \ge 0$.*
2. *How far does the car travel in the time it takes to come to rest?*

Everything we said about velocity, position, and displacement carries over to more general situations. Suppose you are interested in some quantity Q that changes over time: Q may represent the amount of water in a reservoir, the population of a cell culture, or the amount of a resource that is consumed or produced. If you are given the rate Q' at which Q changes, then integration allows you to calculate either the net change in the quantity Q or the future value of Q:

the **net change** in Q between $t = a$ and $t = b > a$ is

$$Q(b) - Q(a) = \int_a^b Q'(t)dt.$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x)dx.$$

Example 4 (§6.1 Ex. 43)**.** *A culture of bacteria in a Petri dish has an initial population of 1500 cells and grows at a rate (in cells/day) of* $N'(t) = 100e^{-0.25t}$*. Assume* t *is measured in days.*

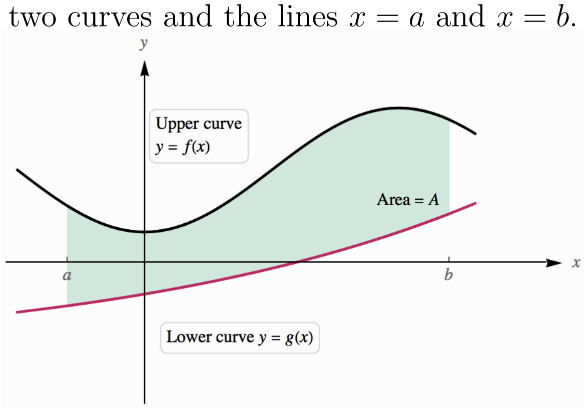
1. *What is the population after 20 days? After 40 days?*

2. *Find the population* $N(t)$ *at any time* $t \geq 0$.

3 Regions between curves

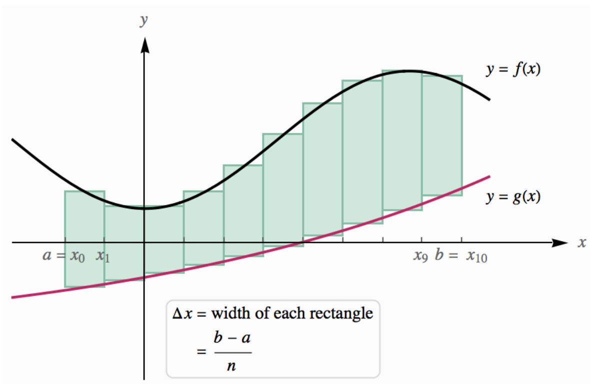
Briggs-Cochran-Gillett-Schulz §6.2 pp. 416 - 418

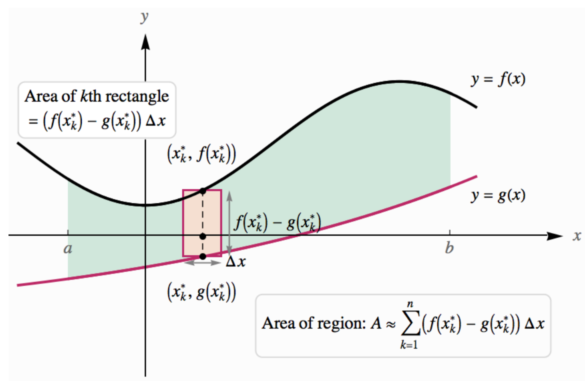
Here we generalize the method for finding the area of a region bounded by a single curve to regions bounded by two or more curves. Consider two functions f and g continuous on an interval $[a, b]$ on which $f(x) \geq g(x)$. The goal is to find the area of the region bounded by the two curves and the lines $x = a$ and $x = b$.



We compute the area using Riemann sums: partition the interval $[a, b]$ into n subintervals using uniformly spaced grid points separated by a distance $\Delta x = (b - a)/n$. On each subinterval, we draw a rectangle extending from the lower curve to the upper curve. On the kth subinterval, we choose a point x_k^*, and the height of the corresponding rectangle is $f(x_k^*) - g(x_k^*)$. Therefore, the area of the kth rectangle is $(f(x_k^*) - g(x_k^*))\Delta x$. Summing the areas of the n rectangles gives an approximation to the area of the region between the curves:

$$A \approx \sum_{k=1}^{n} (f(x_k^*) - g(x_k^*))\Delta x.$$

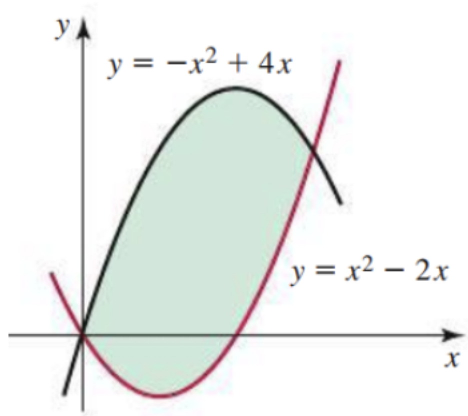


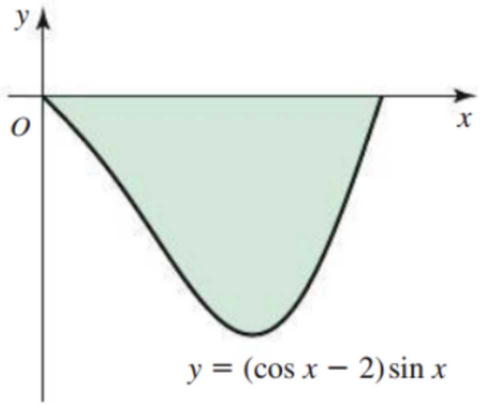


As the number of grid points increases, Δx approaches zero, and these sums approach the area of the region between the curves:

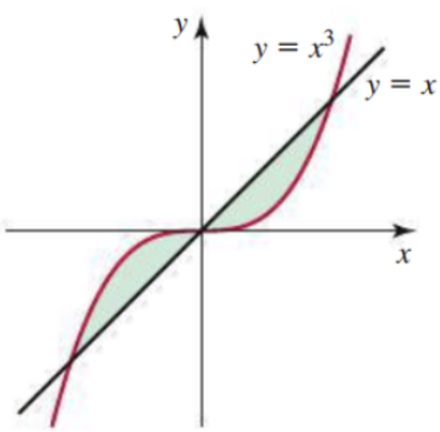
Definition 5. *Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is $A = \int_a^b (f(x) - g(x))dx$.*

Example 6 (§6.2 Ex. 10, 14, 16)**.** *Determine the area of the shaded region in the following figures:*





y
O
x
y = (cos x − 2) sin x



y
y = x³
y = x
x